CLASSIFICATION OF CERTAIN GENERA OF CODES, LATTICES AND VERTEX OPERATOR ALGEBRAS

by

NAKORN JUNLA

M.S., University of Miami, 2007

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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Abstract

We classify the genera of doubly even binary codes, the genera of even lattices, and the genera of rational vertex operator algebras (VOAs) arising from the modular tensor categories (MTCs) of rank up to 4 and central charges up to 16. For the genera of even lattices, there are two types of the genera: code type genera and non code type genera. The number of the code type genera is finite. The genera of the lattices of rank larger than or equal to 17 are non code type. We apply the idea of a vector valued modular form and the representation of the modular group $SL_2(\mathbb{Z})$ in [BG07] to classify the genera of the VOAs arising from the MTCs of ranks up to 4 and central charges up to 16.

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Approved by:

Major Professor
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Chapter 1

Introduction

The notions of the genera of rational vertex operator algebras (VOAs) and the genera of even lattices have been discussed in [Höh03]. There is a relation between the genera of even lattices and VOAs such that a genus of even lattices is a subset of a genus of the corresponding VOAs. It is also known that an even lattice can be constructed from a doubly even binary code (cf. [Ebe02]). By giving a notion of the genera of doubly even binary codes, we have that a genus of doubly even binary codes is a subset of a genus of the corresponding even lattices. So there is an injection from the genera of doubly even codes into the genera of even lattices and there is an injection from the genera of even lattices into the genera of rational VOAs. In this thesis we would like to classify these genera of codes, lattices and VOAs. First, we classify the genera of doubly even binary codes. Then, we classify the genera of even lattices in such a way that they are of code type or not. A code type genus of lattice is a lattice genus which contains only the lattices that can be constructed from a doubly even code. Otherwise, a genus is called a non code type genus. Finally, we classify the genera of the VOAs. We can classify only certain genera of the VOAs. By using the fact that the category of the VOA modules has a structure of a modular tensor category (MTC) [Hua08], we can classify the VOA genera arising from the MTCs of small ranks.

We organize our results into two main chapters: the classification of the genera of code type lattices and the classification of the genera of the rational vertex operator algebras (VOAs) arising from the small modular tensor categories (MTCs).

We give a brief detail for codes and lattices in chapter 2. And we define the genera of doubly even codes, even lattices, and rational VOAs in chapter 3. Let L_C be the integral lattice constructed from the doubly even binary code C. Depending on the length and the dimension of the code C, the genus containing L_C contains either only the lattices constructed from the codes in the same genus as C or there is at least one lattice in the genus that is not constructed from any code. We would like to classify the genera of both types: *code type* and *non code type* genera. By computation, using the computer algebra software such as Magma, we can classify the lattice genera of both types up to the codes of length 23 with at least the largest dimensions. Using the results from the computation and applying some lemmas in chapter 4 and the complete classification of the even unimodular lattices of dimension 24, we can conclude that all the genera associated with the codes of length from 17 with any dimension are non code type. And the code type lattice genera are listed in Proposition 4.2.7. The method of computation and the main results are explained in chapter 4 and Appendix A.

In chapter 5, we classify the genera of the VOAs arising from the small MTCs. The family of the characters of the VOA modules forms a vector valued modular function of a representation ρ of the modular group $SL_2(\mathbb{Z})$ (cf. [Zhu96]). So we have a space consisting of these vector valued modular forms. We apply the idea of the fundamental matrix of the representation of the modular group in [BG07] to classify the space of vector valued modular forms and hence the genera of the VOAs. With the fact that the category of the VOA modules forms a MTC, a genus of the VOAs depend only on the corresponding MTC and the central charge. We use the list of the MTCs classified in [RSW09] in our genera classification. We only study the cases of unitary MTCs and there are a total of 35 of them. This classification is done by computation mainly with Mathematica and Magma. The method of computation and the results are in chapter 5 and in Appendix B.

Chapter 2

Codes, Lattices, Vertex Operator Algebras, and Modular Tensor Categories

2.1 Codes

In this section we introduce the definitions and some properties of binary codes as in [Ebe02] Let $\mathbb{F}_q^n$ be a finite field with $q = p^r$ (p prime).

Definition 2.1.1. A code C of length n is a nonempty proper subset of $\mathbb{F}_q^n$.

If $|C| = 1$ the code is called trivial. If $q = 2$ the code is called a binary code. The elements of C are called codewords, and n is called the wordlength of C.

Let $x = (x_1, ..., x_n) \in \mathbb{F}_q^n$. The weight $w(x)$ of x is the number of nonzero x_i. If $x \in \mathbb{F}_q^n$, $y \in \mathbb{F}_q^n$, then the (Hamming) distance $d(x, y)$ of x and y is defined by $d(x, y) := w(x - y)$.

Let C be a nontrivial code. The minimum of the distance $d(x, y)$ for $x, y \in C$, $x \neq y$, is called the minimum distance of the code C. An (n, M, d)-code is a code with wordlength n, M codewords, and minimum distance d.

Definition 2.1.2. A linear code C is a linear subspace of $\mathbb{F}_q^n$.

If C is a linear code, k is the dimension of C as an $\mathbb{F}_q^n$-vector space and d is its minimum distance, then C is called an $[n, k, d]$-code.

For a linear code C, the minimum distance is equal to the minimum weight, i.e., to the minimum of the weights of non zero codewords.

Consider the vectors of $\mathbb{F}_q^n$ as column vectors. Then a linear code is defined by an exact sequence

$$0 \to \mathbb{F}_q^k \xrightarrow{A} \mathbb{F}_q^n \xrightarrow{B} \mathbb{F}_q^{n-k} \to 0$$

where A and B are linear mappings. The exactness of the sequence is equivalent to the three conditions: rank $A = k, BA = 0$, and rank $B = n - k$. The code C defined by this sequence can be obtained in two ways.

First $C = A(\mathbb{F}_q^k) \subset \mathbb{F}_q^n$. The linear mapping A is given by an $n \times k$ matrix A. The columns of A form a basis of C. Usually one considers the transpose $G = A^t$ of A; this is $k \times n$ matrix for which the rows form a basis of C. G is called a generator matrix of C.

On the other hand $C = \ker B$, i.e., $x \in C$ if and only if $Bx = 0$. The linear mapping B is given by an $(n - k) \times n$ matrix B. The rows of B are the relations defining C. The matrix B is called a parity check matrix of C. For every $x \in \mathbb{F}_q^n$, we call $Bx \in \mathbb{F}_q^{n-k}$ the syndrome of x. The codewords of C are characterized by having syndrome 0.

Let C be a linear code defined by an exact sequence as above. From linear algebra we know that a linear mapping $\phi : V \to W$ between vector spaces V and W induces a dual mapping $\phi^* : W^* \to V^*$ between the corresponding dual spaces W^* and V^*; if V and W are finite dimensional, then we can identify the vector spaces with their corresponding dual spaces after the choice of bases. Therefore the above sequence induces a dual sequence

$$0 \to \mathbb{F}_q^{n-k} \xrightarrow{B^t} \mathbb{F}_q^n \xrightarrow{A^t} \mathbb{F}_q^k \to 0.$$

The condition $BA = 0$ is equivalent to the condition $A^t B^t = 0$. This exact sequence defines the dual code $C^\perp$, i.e., $C^\perp := B^t(\mathbb{F}_q^{n-k})$. If C has dimension k then $C^\perp$ has dimension $n-k$.

For $x, y \in \mathbb{F}_q^n$ we define their scalar product $x \cdot y$ by

$$x \cdot y := \sum_{i=1}^{n} x_i y_i.$$

Lemma 2.1.3 (cf. [Ebe02])**.** *$C^\perp = \{y \in \mathbb{F}_q^n \mid x \cdot y = 0$ for all $x \in C\}$.*

A linear code C is called *self-dual* if and only if $C = C^\perp$. Note that dim C + dim $C^\perp$ $= n$, so $C = C^\perp$ implies that n is even, dim $C = \dfrac{n}{2}$ and $C \subset C^\perp$.

A binary code C is called *doubly even*, if the weights $w(x)$ of all codewords $x \in C$ are divisible by 4. A doubly even code C satisfies $C \subset C^\perp$, since over $\mathbb{Z}$

$$x \cdot y = \frac{1}{2}((x+y)^2 - x^2 - y^2), \text{ where } x^2 = x \cdot x.$$

Now we will give some examples of doubly even codes which are constructed in [DFG+11]. For each N, there is a trivial doubly even code $\{0000\}$ generated by an empty matrix which we call t_N. For each even $N \geq 4$, there is a doubly even code called d_N of length N and with $\frac{N}{2} - 1$ generators, with the generating set

$$\begin{bmatrix} 111100000\cdots00000 \\ 001111000\cdots00000 \\ 000011110\cdots00000 \\ \vdots \\ 000000000\cdots01111 \end{bmatrix}.$$

For example, d_4 is generated by $[1111]$, and d_6 is generated by $\begin{bmatrix} 111100 \\ 001111 \end{bmatrix}$.

When N is congruent to 7 or 8 modulo 8 there is an important doubly even code called e_N, the generating set of which is that of d_N (or $t_1 \oplus d_{N-1}$ when $N \equiv 7 \pmod 8$) augmented by an additional generator of the form 101010.... For example,

$$e_7 : \begin{bmatrix} 1111000 \\ 0011110 \\ 1010101 \end{bmatrix}, \quad e_8 : \begin{bmatrix} 11110000 \\ 00111100 \\ 00001111 \\ 10101010 \end{bmatrix}.$$

e_7 is known as the Hamming code (7,3) and e_8 is the extended Hamming code (8,4).

For any $N \equiv 0 \pmod 4$ there is an $(N,1)$ doubly even code h_N with the generating set [111...1]. Note that $h_4 = d_4$, but $h_N \subset d_N$ for $N =$ 8,12,16,... .

2.2 Lattices

In this section we introduce the definitions and some properties of integral lattices as in [Ebe02].

Definition 2.2.1. A lattice in $\mathbb{R}^n$ is a subset $\Gamma \subset \mathbb{R}^n$ with the property that there exists a basis $(e_1, ..., e_n)$ of $\mathbb{R}^n$ such that $\Gamma = \mathbb{Z}e_1 \oplus ... \oplus \mathbb{Z}e_n$, i.e., Γ consists of all integral linear combinations of the vectors $e_1, ..., e_n$.

Let Γ be a lattice in $\mathbb{R}^n$. A basis $(e_1, ..., e_n)$ of $\mathbb{R}^n$ with $\Gamma = \mathbb{Z}e_1 \oplus ... \oplus \mathbb{Z}e_n$ is called a basis of Γ. The quotient $\mathbb{R}^n/\Gamma$ is an n-dimensional torus. It is obtained by identifying the faces of the fundamental parallelotope

$$P = \{\lambda_1 e_1 + ... + \lambda_n e_n | 0 \leq \lambda_i \leq 1\}.$$

The volume of a lattice is

$$\mathrm{vol}(\mathbb{R}^n/\Gamma) = \mathrm{vol}(P) = |\det((e_1, ..., e_n))|$$

where $((e_1, ..., e_n))$ is the matrix whose rows are the vectors $e_1, ..., e_n$.

More generally, let $\Gamma' \subset \mathbb{R}^n$ be a lattice with $\Gamma' \subset \Gamma$. Then clearly the index $|\Gamma/\Gamma'|$ is finite and

$$\mathrm{vol}(\mathbb{R}^n/\Gamma') = \mathrm{vol}(\mathbb{R}^n/\Gamma)|\Gamma/\Gamma'|.$$

We denote the Euclidean scalar product of two vectors $x, y \in \mathbb{R}^n$ by $x \cdot y$. So

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

The Euclidean scalar product is a non-degenerate, positive definite, symmetric bilinear form. Put $a_{ij} = e_i \cdot e_j$ and let A be the matrix $((a_{ij}))$. Let C be the matrix $((e_1, ...e_n))$. Then $A = CC^t$. Therefore,

$$\mathrm{vol}(P) = |\det C| = \sqrt{\det A}.$$

Let $V = \mathbb{R}^n$. We identify V with the dual vector space $V^* = \mathrm{Hom}(V, \mathbb{R})$ by means of the mapping $V \to V^*, x \mapsto f_x$, with $f_x(y) = x \cdot y$. Let Γ be a lattice in $\mathbb{R}^n$. We denote the dual lattice of Γ by Γ^*. It is

$$\Gamma^* = \mathrm{Hom}(\Gamma, \mathbb{Z}) = \{x \in \mathbb{R}^n | x \cdot y \in \mathbb{Z} \text{ for all } y \in \Gamma\}.$$

Let $(e_1, ..., e_n)$ be a basis of Γ, and let $(e_1^*, ..., e_n^*)$ be the dual basis , i.e., $e_i^* \cdot e_j = \delta_{ij}$. Then $e_i^* = \sum_{j=1}^{n} b_{ij} e_j$ and $B = ((b_{ij})) = A^{-1}$. The e_i^* form a basis of Γ^*.

A lattice $\Gamma \in \mathbb{R}^n$ is called *integral*, if $x \cdot y \in \mathbb{Z}$ for all $x, y \in \Gamma$.

A lattice $\Gamma \in \mathbb{R}^n$ is called *unimodular* if $\Gamma^* = \Gamma$.

Now let Γ be an integral lattice with basis $(e_1, ..., e_n)$, and let A be the matrix $A =$

$((e_i \cdot e_j))$. Then A is an integral matrix and the determinant $\det A$ of A is an integer, and it is called the *discriminant* of the lattice Γ, written $\mathrm{disc}(\Gamma)$. And so

$$\mathrm{disc}(\Gamma) = |\Gamma^*/\Gamma|.$$

Let Γ be an integral lattice in $\mathbb{R}^n$. A $\mathbb{Z}$-submodule Λ of Γ is called a *sublattice* of Γ. It is a lattice in some subspace $W \subset \mathbb{R}^n$ which is isomorphic to $\mathbb{R}^k$ for some k. In particular, the dual lattice Λ^* is defined to be

$$\Lambda^* = \{x \in W | x \cdot y \in \mathbb{Z} \text{ for all } y \in \Lambda\}.$$

A sublattice Λ of Γ is call primitive if Γ/Λ is a free $\mathbb{Z}$-module. If K is a subset of Γ we call the $\mathbb{Z}$-submodule $K^\perp = \{y \in \Gamma | x \cdot y = 0 \text{ for all } x \in K\}$ the sublattice orthogonal to K.

Let $\Lambda_1, ..., \Lambda_m$ be sublattices of Γ. The lattice Γ is called the orthogonal direct sum of the sublattices $\Lambda_1, ..., \Lambda_m$ denoted by $\Gamma = \Lambda_1 \oplus ... \oplus \Lambda_m$, if Γ is the direct sum of the submodules $\Lambda_1, ..., \Lambda_m$ and $x \cdot y = 0$ for all $x \in \Lambda_i, y \in \Lambda_j$, and $i \neq j$.

Definition 2.2.2. An integral lattice Γ is called *even* if $x^2 = x \cdot x \equiv 0 (\mathrm{mod}\ 2)$ for all $x \in \Gamma$.

In matrix terms, this means that the diagonal elements $e_i \cdot e_j$ of the matrix A are all even.

Let L be an even lattice in $\mathbb{R}^n$. Then we have a canonical embedding $L \hookrightarrow L^*$ into the dual lattice of L. The quotient group

$$A := L^*/L$$

is a finite abelian group of order $\mathrm{disc}(L)$. We define a mapping $b_A : A \times A \to \mathbb{Q}/\mathbb{Z}$ by

$$b_A(x + L, y + L) = x \cdot y + \mathbb{Z}, \text{ where } x, y \in L^*,$$

and a mapping $q_A : A \to \mathbb{Q}/2\mathbb{Z}$, by

$$q_A(x+L) = x^2 + 2\mathbb{Z}, \text{ where } x \in L^*.$$

Then b_A is a finite symmetric bilinear form, and q_A is a *finite quadratic form*. By this we mean a mapping $q : G \to \mathbb{Q}/2\mathbb{Z}$ defined on a finite abelian group G satisfying the following conditions:

(i) $q(rx) = r^2 q(x)$ for all $r \in \mathbb{Z}$ and $x \in A$,

(ii) $q(x+y) - q(x) - q(y) \equiv 2b(x,y) (\bmod 2\mathbb{Z})$,

where $b : G \times G \to \mathbb{Q}/\mathbb{Z}$ is a finite symmetric bilinear form, which we call the bilinear form corresponding to q. The form q_A is called the *discriminant quadratic form* of L.

Next we will introduce the definition of the root lattices.

Let $L \subset \mathbb{R}^n$ be an even lattice, i.e., $x^2 \in 2\mathbb{Z}$ for all $x \in L$. Let

$$R := \{x \in L | x^2 = 2\}.$$

An element $x \in R$ is called a root.

Definition 2.2.3. An even lattice $L \subset \mathbb{R}^n$ is called a *root lattice*, if R generates L.

A lattice L is called *reducible*, if L is the orthogonal direct sum $L = L_1 \oplus L_2$ of two lattices $L_1 \subset \mathbb{R}^{n_1}, L_2 \subset \mathbb{R}^{n_2}$ with $n_1, n_2 \geq 1$; otherwise it is called *irreducible*.

Theorem 2.2.4 (cf. [Ebe02])**.** *Every root lattice is the orthogonal direct sum of irreducible root lattices.*

There are five types of irreducible root lattices: Types A_n, D_n $(n \geq 3)$, E_6, E_7, and E_8 (cf. Section 1.4 in [Ebe02]).

Let $L \subset \mathbb{R}^n$ be a lattice. We associate to L a function which is defined on the upper half plane

$$\mathbb{H} = \{\tau \in \mathbb{C} | \mathrm{Im}\tau > 0\} \subset \mathbb{C}.$$

For $\tau \in \mathbb{H}$ let $q = e^{2\pi i \tau}$

Definition 2.2.5. The *theta function* of the lattice L is the function

$$\theta_L(\tau) := \sum_{x \in L} q^{\frac{1}{2} x \cdot x}$$

for $\tau \in \mathbb{H}$.

2.3 Vertex Operator Algebras (VOAs)

In this section we introduce the definitions and some properties of vertex operator algebras as in [FBZ04].

2.3.1 Formal distribution

Let $\mathbf{R}$ be a $\mathbb{C}$-algebra.

Definition 2.3.1. An $\mathbf{R}$-valued formal power series (or formal distribution) in variables $z_1, z_2, ..., z_n$ is an arbitrary (finite or infinite) series

$$A(z_1, z_2, ..., z_n) = \sum_{i_1 \in \mathbb{Z}} \cdots \sum_{i_n \in \mathbb{Z}} A_{i_1, ..., i_n} z_1^{i_1} \cdots z_n^{i_n}, \tag{2.3.1}$$

where each $A_{i_1,...,i_n} \in \mathbf{R}$. These series form a vector space, which is denoted by $\mathbf{R}[[z_1^{\pm 1}, ..., z_n^{\pm 1}]]$.

In general, a product of two elements of $\mathbf{R}[[z_1^{\pm 1}, ..., z_n^{\pm 1}]]$ does not make sense, since individual coefficients of the product are infinite sums of coefficients of the factors. However, the product of a formal power series by a Laurent polynomial (i.e., a series (2.3.1) such that $A_{i_1,...,i_n} = 0$ for all but finitely many n-tuples $(i_1, ..., i_n)$ is always well-defined.

Definition 2.3.2. Given a formal power series in one variable, $f(z) = \sum_{z\in\mathbb{Z}} a_i z^i$, we define its residue (at 0) as

$$\mathrm{Res} f(z)dz = \mathrm{Res}_{z=0} f(z)dz = a_{-1}.$$

Note that if $\mathbf{R} = \mathbb{C}$ and $f(z)$ is the Laurent series of a meromorphic function defined on a disc around 0, having poles only at 0, then

$$\mathrm{Res}_{z=0} f(z)dz = \frac{1}{2\pi i}\oint f(z)dz,$$

where the integral is taken over a closed curve winding once around 0.

Any formal power series $f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$ in $\mathbb{C}[[z^{\pm 1}]]$ defines a linear functional on the space of Laurent polynomials $\mathbb{C}[z, z^{-1}]$ whose value on $g \in \mathbb{C}[z, z^{-1}]$ equals

$$\mathrm{Res}_{z=0} f(z)g(z)dz.$$

Definition 2.3.3. The *formal delta-function* $\delta(z-w)$ is a formal power series in two variables z, w defined by

$$\delta(z-w) = \sum_{m\in\mathbb{Z}} z^m w^{-m-1}. \tag{2.3.2}$$

Its coefficients $a_{mn} = \delta_{m,-n-1}$ are supported on the diagonal $m + n = -1$, and hence it can be multiplied by an arbitrary formal power series in one variable (i.e., depending only on z or only on w). Indeed, for such a series $A(w)$, we obtain

$$A(w)\delta(z-w) = \sum_{k\in\mathbb{Z}} A_k w^k \sum_{m\in\mathbb{Z}} z^m w^{-m-1} = \sum_{m,n\in\mathbb{Z}} A_{m+n+1} z^m w^n,$$

so each coefficient is well-defined. Furthermore, the formula above shows that as formal power series in z, w,

$$A(z)\delta(z-w) = A(w)\delta(z-w), \tag{2.3.3}$$

which motivates the terminology “delta-function”.

We obtain from formula (2.3.3) that

$$(z-w)\delta(z-w) = 0 \tag{2.3.4}$$

and, by induction,

$$(z-w)^{n+1}\partial_w^n \delta(z-w) = 0. \tag{2.3.5}$$

Lemma 2.3.4 (cf. [FBZ04])**.** *Let $f(z,w)$ be a formal power series in $\boldsymbol{R}[[z^{\pm 1}, w^{\pm 1}]]$ satisfying $(z-w)^N f(z,w) = 0$ for a positive integer N. Then $f(z,w)$ can be written uniquely as a sum*

$$\sum_{i=0}^{N-1} g_i(w)\partial_w^i \delta(z-w),\ \ g_i(w) \in \boldsymbol{R}[[w^{\pm 1}]]. \tag{2.3.6}$$

2.3.2 Fields

Definition 2.3.5. [Fields] Let V be a vector space over $\mathbb{C}$. Denote by $\mathrm{End}V$ the algebra of linear operators on V. A formal power series

$$A(z) = \sum_{j\in\mathbb{Z}} A_j z^{-j} \in \mathrm{End}V[[z^{\pm 1}]] \tag{2.3.7}$$

is called a *field* on V if for any $v \in V$ we have $A_j \cdot v = 0$ for large enough j.

In other words, $A(z)\cdot v$ is an element of $V((z))$, the space of formal Laurent series with coefficients in V (i.e., it has only finitely many terms with negative powers of z). Fields on

V form a vector space denoted by $\mathcal{F}(V)$.

For any $\mathbb{C}$-algebra $\mathbf{R}$, we denote by $\mathbf{R}[[z]]$ the space of $\mathbf{R}$-valued formal Taylor series in z. The space $\mathbf{R}((z))$ of $\mathbf{R}$-valued formal Laurent series in z is by definition the space of series $\sum_{n\in\mathbb{Z}} a_n z^n$, where $a_n \in \mathbf{R}$ for all n, and there exists $N \in \mathbb{Z}$ such that $a_n = 0, \forall n \leq N$. Note that $\mathbf{R}((z))$ is an algebra.

Denoted by $\mathbb{C}((z))((w))$ the space $\mathbf{R}((w))$, where $\mathbf{R} = \mathbb{C}((z))$. In other words, this is the space of Laurent series in w whose coefficients are Laurent series in z.

For any vector $v \in V$ and any linear functional $\varphi : V \to \mathbb{C}$, the matrix element $\langle \varphi, A(z)v\rangle$ of a field $A(z)$ is a Laurent power series.

Given another field, $B(w)$, we consider the composition $A(z)B(w)$ as an End$V$-valued formal power series in z, w. Given $v \in V$ and $\varphi \in V^*$ (where V^* denotes the vector space of all linear functionals on V), consider the matrix element

$$\langle \varphi, A(z)B(w)v\rangle \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]].$$

From the definition of a field , we see that this formal power series actually belongs to $\mathbb{C}((z))((w))$.

Definition 2.3.6. Two fields $A(z)$ and $B(w)$ acting on a vector space V are said to be *local* with respect to each other if for every $v \in V$ and $\varphi \in V^*$, the matrix elements

$$\langle \varphi, A(z)B(w)v\rangle \text{ and } \langle \varphi, B(w)A(z)v\rangle$$

are expansions of one and the same element

$$f_{v,\varphi} \in \mathbb{C}[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$$

in $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$, respectively, and the order of pole of $f_{v,\varphi}$ in $(z - w)$ is uniformly bounded for all v, φ.

The last condition above may be reformulated as saying that there exists $N \in \mathbb{Z}_+$ such that

$$(z-w)^N f_{v,\varphi} \ \in \mathbb{C}[[z,w]][z^{-1},w^{-1}]$$

for all v, φ. But then the expansions of $(z-w)^N f_{v,\varphi}$ in $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$ are equal to each other. Therefore if $A(z)$ and $B(w)$ are local with respect to each other, then

$$(z-w)^N A(z)B(w) = (z-w)^N B(w)A(z),$$

or equivalently, $(z-w)^N[A(z),B(w)] = 0$, where $[A,B] := AB - BA$. The following proposition shows that the converse is also true.

Proposition 2.3.7 (cf. [FBZ04]). *Two fields $A(z), B(w)$ are local if and only if there exists $N \in \mathbb{Z}_+$ such that*

$$(z-w)^N[A(z),B(w)] = 0 \tag{2.3.8}$$

as a formal power series in $\mathrm{End}V[[z^{\pm 1}, w^{\pm 1}]]$.

2.3.3 Definition of a Vertex Algebra

Definition 2.3.8. A vertex algebra is a collection of data:

- (space of states) a vector space V;
- (vacuum vector) a vector $|0\rangle \in V$;
- (translation operator) a linear operator $T : V \to V$;
- (vertex operators) a linear operation

$$Y(\cdot, z) : V \to \mathrm{End}V[[z^{\pm 1}]]$$

taking each $A \in V$ to a field acting on V,

$$Y(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}.$$

These data are subject to the following axioms:

- (vacuum axiom) $Y(|0\rangle, z) = \mathrm{Id}_V$. Furthermore, for any $A \in V$ we have

  $$Y(A, z)|0\rangle \in V[[z]].$$

  so that $Y(A, z)|0\rangle$ has a well-defined value at $z = 0$, and

  $$Y(A, z)|0\rangle|_{z=0} = A.$$

  In other words, $A_{(n)}|0\rangle = 0, n \geq 0$, and $A_{(-1)}|0\rangle = A$.

- (translation axiom) For any $A \in V$,

  $$[T, Y(A, z)] = \partial_z Y(A, z)$$

  and $T|0\rangle = 0$.

- (locality axiom) All fields $Y(A, z)$ are local with respect to each other.

A vertex algebra is called $\mathbb{Z}$-graded if V is a $\mathbb{Z}$-graded vector space, $|0\rangle$ is a vector of degree 0, T is a linear operator of degree 1, and for $A \in V_m$ the field $Y(A, z)$ has conformal dimension m (i.e., $\deg A_{(n)} = -n + m - 1$).

Definition 2.3.9. A *vertex algebra homomorphism* ρ between vertex algebras

$$(V, |0\rangle), T, Y) \to (V', |0\rangle', T', Y')$$

is a linear map $V \to V'$ mapping $|0\rangle$ to $|0\rangle'$, intertwining the translation operators, and satisfying

$$\rho(Y(A,z)B) = Y(\rho(A),z)\rho(B).$$

A *vertex subalgebra* $V' \subset V$ is a T-invariant subspace containing the vacuum vector, and satisfying $Y(A,z)B \in V'((z))$ for all $A, B \in V'$ (with the induced vertex algebra structure).

A *vertex algebra ideal* $I \subset V$ is a T-invariant subspace satisfying $Y(A,z)B \in I((z))$ for all $A \in I$ and $B \in V$. And by the skew-symmetry property, we have $Y(B,z)A \in I((z))$ as well. It follows that for any proper ideal $I, V/I$ inherits a natural quotient vertex algebra structure.

Lemma 2.3.10 (cf. [FBZ04]). *For two vertex algebras* $(V_1, |0\rangle_1, T_1, Y_1)$ *and* $(V_2, |0\rangle_2, T_2, Y_2)$, *the data* $(V_1 \otimes_{\mathbb{C}} V_2, |0\rangle_1 \otimes |0\rangle_2, T_1 \otimes 1 + 1 \otimes T_2, Y)$, *where*

$$Y(A_1 \otimes A_2, z) = Y_1(A_1, z) \otimes Y_2(A_2, z)$$

defines a vertex algebra called the tensor product of V_1 *and* V_2.

2.3.4 Examples of Vertex Algebras

There are some examples of vertex algebras in [FBZ04] such as the vertex algebra associated to the Heisenberg Lie algebra which define the vertex algebra structure via the Fock representation π, the vertex algebra associated to the Affine Kac-Moody algebras, and the Virasoro vertex algebra.

We will give brief details of the affine Kac-Moody algebras and their vertex algebras and the Virasoro vertex algebra below (see [FBZ04] for the full details).

An **affine Kac-Moody algebra** is defined as a central extension of the formal loop algebra. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. We define the formal *loop algebra* of $\mathfrak{g}$,

$$L\mathfrak{g} = \mathfrak{g}((t)) = \mathfrak{g} \otimes \mathbb{C}((t)),$$

as the Lie algebra with the commutator

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t).$$

We now define the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ as a central extension

$$0 \to \mathbb{C}K \to \hat{\mathfrak{g}} \to L\mathfrak{g} \to 0.$$

As a vector space, $\hat{\mathfrak{g}} \simeq L\mathfrak{g} \oplus \mathbb{C}K$, with the commutation relation $[K, \cdot] = 0$ (so K is central) and

$$[A \otimes f(t), B \otimes g(t)] = [A, B]f(t)g(t) - (\mathrm{Res}_{t=0} f dg)(A, B)K.$$

The Kac-Moody cocycle is non-trivial, i.e., $\mathfrak{g}$ cannot be split as a Lie algebra into a direct sum $L\mathfrak{g} \oplus \mathbb{C}K$. Thus the Kac-Moody extension is a universal central extension of $L\mathfrak{g}$, i.e., any other central extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ admits a Lie algebra homomorphism $\hat{\mathfrak{g}} \to \tilde{\mathfrak{g}}$.

The vacuum representation. Inside the loop algebra $L\mathfrak{g} = \mathfrak{g}((t))$ there is a "positive" Lie subalgebra $\mathfrak{g}[[t]] = \mathfrak{g} \otimes \mathbb{C}[[t]]$. If $f, g \in \mathbb{C}[[t]]$, then $\mathrm{Res}_{t=0} f dg = 0$. Hence the central extension becomes trivial when restricted to this subspace, and so $\mathfrak{g}[[t]]$ and $\mathfrak{g} \oplus \mathbb{C}K$ are Lie subalgebra of $\hat{\mathfrak{g}}$.

Now consider the one-dimensional representation $\mathbb{C}_k$ of $\mathfrak{g}[[t]] \oplus \mathbb{C}K$ on which $\mathfrak{g}[[t]]$ acts by 0 and K acts as multiplication by a scalar $k \in \mathbb{C}$. We define the *vacuum representation of level* k of $\hat{\mathfrak{g}}$ as the representation induced from $\mathbb{C}_k$:

$$V_k(\mathfrak{g}) = \mathrm{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g}[[t]] \oplus \mathbb{C}K} \mathbb{C}K = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where $U(\hat{\mathfrak{g}})$ denotes the universal enveloping algebra of $\hat{\mathfrak{g}}$. More generally, we will say that a module M over $\hat{\mathfrak{g}}$ has level $k \in \mathbb{C}$, if K acts on M as multiplication by k.

Vertex algebra structure. We now can define a vertex algebra structure on the vacuum representation. Let $\{J^a\}_{a=1,\ldots,\dim \mathfrak{g}}$ be an ordered basis of $\mathfrak{g}$. Split the extension $\hat{\mathfrak{g}}$

as a vector space. For any $A \in \mathfrak{g}$ and $n \in \mathbb{Z}$, we denote

$$A_n \stackrel{\text{def}}{=} A \otimes t^n \in L\mathfrak{g}.$$

Then the elements $J^a_n, n \in \mathbb{Z}$, and $\mathbf{K}$ form a (topological) basis for $\hat{\mathfrak{g}}$, while the elements $J^a_n, n \geq 0$, and $\mathbf{K}$ form a basis for the "positive" subalgebra from which we induced $V_k(\mathfrak{g})$ has a PBW basis of monomials of the form

$$J^{a_1}_{n_1} ... J^{a_m}_{n_m} v_k,$$

where $n_1 \leq n_2 \leq ... \leq n_m < 0$, and if $n_i = n_{i+1}$, then $a_i = a_{i+1}$.

Definition 2.3.11. The *normally ordered product* of the fields

$$A(z) = \textstyle\sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1},\ B(w) = \sum_{m\in\mathbb{Z}} B_{(m)} w^{-m-1}$$

is defined as the formal power series

$$\begin{aligned} : A(z)B(w) : \quad &= \textstyle\sum_{n\in\mathbb{Z}} \left(\sum_{m<0} A_{(m)} B_{(n)} z^{-m-1} + \sum_{m\geq 0} B_{(n)} A_{(m)} z^{-m-1}\right) w^{-n-1} \\ &= A(z)_+ B_(w) + B(w) A(z)_-, \end{aligned}$$

where for a formal power series $f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$, we write

$$f(z)_+ = \textstyle\sum_{n\geq 0} f_n z^n,\ f(z)_- = \sum_{n<0} f_n z^n.$$

We define a $\mathbb{Z}_+$-graded vertex algebra structure on $V_k(\mathfrak{g})$ as follows:

- Vacuum vector : $|0\rangle = v_k$.
- Translation operator : $Tv_k = 0$, $[T, J^a_n] = -nJ^a_{n-1}$.
- Vertex operators : $Y(v_k, z) = \text{Id}$,

$$Y(J^a_{-1} v_k, z) = J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1},$$

and in general,

$$Y(J^{a_1}_{n_1}...J^{a_m}_{n_m}v_k, z) = \frac{1}{(-n_1-1)!...(-n_m-1)!} : \partial_z^{-n_1-1}J^{a_1}(z)...\partial_z^{-n_m-1}J^{a_m}(z) : .$$

- $\mathbb{Z}_+$-gradation :

$$\deg\ J^{a_1}_{n_1}...J^{a_m}_{n_m}v_k = -\sum_{i=1}^{m} n_i.$$

Next we will define the **Virasoro vertex algebra**. Let $K = \mathbb{C}((t))$ and $O = \mathbb{C}[[t]]$. Consider the Lie algebra $\mathrm{Der}K = \mathbb{C}((t))\partial_t$ of derivation of K. The Virasoro algebra is by definition the central extension of $\mathrm{Der}K$:

$$0 \to \mathbb{C}C \to Vir \to \mathrm{DerK} \to 0,$$

defined by the commutation relations

$$[f(t)\partial_t, g(t)\partial_t] = (fg' - f'g)\partial_t - \frac{1}{12}(\mathrm{Res}_{t=0} fg''' dt)C.$$

It is known that this extension is universal. It has generators C, and

$$L_n = -t^{n+1}\partial_t,\ n \in \mathbb{Z},$$

satisfying the relations that C is central and

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}C.$$

We will say that a module M over the Virasoro algebra has *central charge* $c \in \mathbb{C}$, if C acts on M by multiplication by c.

Now we are ready to define the Virasoro vertex algebra. Note that $\mathrm{Der}O = \mathbb{C}[[t]]\partial_t$. We can pick the induced representation in which the generating vector $|0\rangle = 0$ for all $n \geq -1$.

More precisely, let $U(Vir)$ be the universal enveloping algebra of Vir. For each $c \in \mathbb{C}$ we define the induced representation

$$\mathrm{Vir}_c = \mathrm{Ind}^{Vir}_{\mathrm{Der}O \oplus \mathbb{C}C} \mathbb{C}_C = U(Vir) \otimes_{U(\mathrm{Der}O \oplus \mathbb{C}C)} \mathbb{C}_c,$$

where C acts as multiplication by c and $\mathrm{Der}O$ acts by zero on the one-dimensional module $\mathbb{C}_C$. Note that Vir_C has central charge c as a module over the Virasoro algebra.

By the Poincaré-Birkhoff-Witt theorem, Vir_c has a PBW basis consisting of monomials of the form

$$L_{j_1}...L_{j_m}v_c,\ j_1 \leq j_2 \leq ... \leq j_m \leq -2. \tag{2.3.9}$$

Here v_C is the image of $1 \otimes 1 \in U(Vir) \otimes \mathbb{C}_C$ in the induced representation, and we take it to be the vacuum vector of the vertex algebra. We define a $\mathbb{Z}_+$-gradation on Vir_C by the formulas $\mathrm{deg} L_n = -n$, $\mathrm{deg} v_C = 0$.

As the translation operator we take $T = L_{-1}$ and set

$$Y(L_{-2}v_C, z) \stackrel{def}{=} T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

This is the generating field of Vir_C. It has conformal dimension 2. Next we define the vertex operators $Y(A, z)$ for the PBW monomial of the form 2.3.9 :

$$Y(L_{j_1}...L_{j_m}v_C, z) = \frac{1}{(-j_1-2)!}...\frac{1}{(-j_m-2)!} : \partial_z^{-j_1-2}T(z)...\partial_z^{-j_m-2}T(z) : .$$

The Virasoro vertex algebra Vir_c is reducible as a module over the Virasoro algebra if and only if

$$c = c(p,q) \stackrel{\mathrm{def}}{=} 1 - \frac{6(p-q)^2}{pq},\ p, q > 1,\ (p, q) = 1.$$

Let $L_{c(p,q)}$ be the irreducible quotient of $Vir_{c(p,q)}$ (cf. [FBZ04]). Then $L_{c(p,q)}$ is a vertex algebra which is a simple quotient of $Vir_{c(p,q)}$.

2.3.5 Further Definitions and Theorems

Definition 2.3.12. [Vertex operator algebra] A $\mathbb{Z}_{\geq 0}$-graded vertex algebra V is called a *vertex operator algebra* (VOA), of central charge $c \in \mathbb{C}$, if we are given a non-zero conformal vector $\omega \in V_2$ such that the Fourier coefficients L_n^V of the corresponding vertex operator

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n^V z^{-n-2}$$

satisfy the definition relations of the Virasoro algebra with central charge c, and in addition we have $L_{-1}^V = T$, $L_0^V|_{V_n} = n\mathrm{Id}$.

The Virasoro vertex algebra Vir_C is clearly a VOA, with central charge c and conformal vector $\omega = L_{-2}v_c$.

Definition 2.3.13. [Modules over vertex algebras] Let $(V, |0\rangle, T, Y)$ be a vertex algebra. A vector space M is called a $V-module$ if it is equipped with an operation $Y_M : V \to \mathrm{End}M[[z^{\pm 1}]]$ which assigns to each $A \in V$ a field

$$Y_M(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)}^M z^{-n-1}$$

on M subject to the following axioms:

- $Y_M(|0\rangle, z) = \mathrm{Id}_M$;
- for all $A, B \in V$, $C \in M$ the three expressions

$$Y_M(A, z)Y_M(B, w)C \in M((Z))((w)),$$

$$Y_M(B, w)Y_M(A, z)C \in M((w))((z)), \text{ and}$$

$$Y_M(Y(A,z-w)B,w)C \in M((w))((z))$$

are the expressions, in their respective domains, of the same element of

$$M[[z,w]][z^{-1},w^{-1},(z-w)^{-1}].$$

If V is $\mathbb{Z}$-graded, then a V-module M is called *graded* if M is a $\mathbb{C}$-graded vector space and for $A \in V_m$ the field $Y_M(A,z)$ has conformal dimension m, i.e., the operator $A^M_{(n)}$ is homogeneous of degree $-n+m-1$.

These axioms imply that V is a module over itself. And we also have the notions of a submodule and quotient module. A module M whose only submodules are 0 and itself is called *simple* or *irreducible*.

Now we will define a *lattice vertex algebra* as follows: Let $\widetilde{\mathcal{H}}$ be the Weyl algebra (cf. Section 2.1.2 in [FBZ04]). For $\lambda \in \mathbb{C}$, let π_λ be the $\widetilde{\mathcal{H}}$ module generated by a vector $|\lambda\rangle$ such that

$$b_n|\lambda\rangle = 0,\ n \geq 0,\ b_0|\lambda\rangle = \lambda|\lambda\rangle.$$

Let L be a lattice of finite rank equipped with a symmetric bilinear form $(\cdot,\cdot) : L \times L \to \mathbb{Z}$ such that $(\lambda,\lambda) > 0$ for all $\lambda \in L \setminus \{0\}$.

Set $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$. The bilinear form on L indicates a bilinear form on $\mathfrak{h}$, for which we use the same notation. Let $\widehat{\mathfrak{h}}$ be the central extension of $\mathfrak{h}((t))$,

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\mathfrak{h}} \to \mathfrak{h}((t)) \to 0,$$

with the commutation relations

$$[A \otimes f(t), B \otimes g(t)] = -(A,B)(\mathrm{Res} f(t)g'(t)dt)\mathbf{1}.$$

Define the Weyl algebra $\widetilde{\mathcal{H}}_L$ as the enveloping algebra of $\widehat{\mathfrak{h}}$ module the relation $\mathbf{1} = 1$.

It has generators $h_n, h \in \mathfrak{h}, n \in \mathbb{Z}$, and relation

$$[h_n, g_m] = n(h,g)\delta_{n,-m}.$$

For $\lambda \in \mathfrak{h}$, define the Fock representation π_λ of $\widetilde{\mathcal{H}}_L$, generated by the vector $|\lambda\rangle$, such that

$$h_n|\lambda\rangle = 0,\ n > 0;\ h_0|\lambda\rangle = (\lambda, h)|\lambda\rangle.$$

The Fock representation π_0 carries a vertex algebra structure, define in the same way as in the case when $\dim\mathfrak{h} = 1$.

Definition 2.3.14. [Rational vertex algebras] A vertex operator algebra V is called *rational* if every $\mathbb{Z}_{\geq 0}$-graded V-module is completely reducible (i.e., isomorphic to a direct sum of simple V-modules).

This condition implies that

1. V has *finitely many* inequivalent simple $\mathbb{Z}_{\geq 0}$-graded modules;
2. the graded components of each simple $\mathbb{Z}_{\geq 0}$-graded V-module are finite dimensional.

If M is a simple $\mathbb{Z}_{\geq 0}$-graded V-module, then the Virasoro operator L_0^M on M is automatically semi-simple and hence defines a gradation on M. Any other $\mathbb{Z}_{\geq 0}$-gradation on M will necessarily coincide with it up to a shift by a complex number. The above properties allow us to attach to a $\mathbb{Z}_{\geq 0}$-graded simple V-module M its *character*

$$\mathrm{ch}\ M = \mathrm{Tr}_M q^{L_0^M - c/24} = \sum_\alpha \dim M_\alpha q^{\alpha - c/24},$$

where M_α is the subspace of M on which L_0^M acts by multiplication by α, c is the central charge of V, and $q = e^{2\pi i \tau}$.

Now let $C_2(V)$ be the subspace of V spanned by all elements of the form $A_{-2} \cdot B$ for all

$A, B \in V$. Then a rational vertex algebra V is said to satisfy the C_2 *cofiniteness condition* if

1. $\dim V/C_2(V) < \infty$;
2. every vector in V can be written as a linear combination of vectors of the form $L_{n_1} ... L_{n_k} A, n_i < 0$, where A satisfies $L_n A = 0$ for all $n > 0$.

Theorem 2.3.15 (cf.Y.Zhu [Zhu96])**.** *Let V be a rational vertex algebra satisfying the C_2 cofiniteness condition, and let $\{M^1, ..., M^n\}$ be the set of all inequivalent simple $\mathbb{Z}$-graded V-modules (up to an isomorphism). Then the vector space spanned by* ch $M^i, i = 1, ..., n$, *is invariant under the action of $SL_2(\mathbb{Z})$.*

The lattice vertex algebra is one of the examples of rational vertex algebras (cf.[Don93]). Let L be an even positive definite lattice in a real vector space W. We can attached to it a vertex algebra V_L. Its inequivalent simple modules are parameterized by L^*/L, where L^* is the dual lattice. The characters of these modules are the theta-functions corresponding to L. The vertex algebra V_L is the chiral symmetry algebra of the free bosonic conformal field theory compactified on the torus W/L.

Definition 2.3.16. (cf. [Höh03]) A VOA V is called unitary if V can be defined over the real numbers and the natural invariant symmetric form on it is positive definite.

The irreducible quotient of $Vir_{c(p,q)}$, $L_{c(p,q)}$ is a rational VOA which is called the "minimal model" of conformal field theory. If $L_{c(p,q)}$ is unitary, i.e., $c(p,q) < 1$ or $q = p + 1$, then $c(p,q) = 1 - \dfrac{6}{p(p+1)}$ for $p = 2, 3, 4, ...$ We call $L_c(0)$ the Virasoro minimal model VOA.

2.4 Modular Tensor Categories (MTCs)

In this section we introduce the definitions and some properties of modular tensor categories as in [Tur94].

2.4.1 Ribbon Categories

Definition 2.4.1. [Monoidal categories] A *strict monoidal category* is a category $\mathcal{C}$ equipped with a tensor product and an object $\mathbf{1} = \mathbf{1}_{\mathcal{C}}$, called the unit object, such that the following conditions hold.
For any object V of $\mathcal{C}$

$$V \otimes \mathbf{1} = V, \ \ \mathbf{1} \otimes V = V \tag{2.4.1}$$

and for any triple U, V, W of objects of $\mathcal{C}$, we have

$$(U \otimes V) \otimes W = U \otimes (V \otimes W). \tag{2.4.2}$$

For any morphism f in $\mathcal{C}$,

$$f \otimes \mathrm{id}_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}} \otimes f = f \tag{2.4.3}$$

and for any triple f, g, h of morphisms in $\mathcal{C}$,

$$(f \otimes g) \otimes h = f \otimes (g \otimes h). \tag{2.4.4}$$

More general monoidal categories are defined similarly to strict monoidal categories though instead of (2.4.1), (2.4.2) one assumes that the right-hand sides and left-hand sides of these equalities are related by fixed isomorphisms. These fixed isomorphisms should satisfy two compatibility conditions called the pentagon and triangle identities. These isomorphisms should also appear in (2.4.3) and (2.4.4) in the obvious way. And we can consider mainly with strict monoidal categories because of MacLane's coherence theorem which establishes equivalence of any monoidal category to a certain strict monoidal category.

The tensor multiplication of modules over a commutative ring is commutative in the sense that for any modules V, W, there is a canonical isomorphism $V \otimes W \to W \otimes V$. This isomorphism transforms any vector $v \otimes w$ into $w \otimes v$ and extends to $V \otimes W$ by linearity. It is called the flip and denoted by $P_{V,W}$. The system of flips is compatible with the tensor product in the obvious way: for any three modules U, V, W, we have

$$P_{U,V\otimes W} = (\mathrm{id}_V \otimes P_{U,W})(P_{U,V} \otimes \mathrm{id}_W),\ \ P_{U\otimes V,W} = (P_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes P_{V,W}).$$

Definition 2.4.2. [Braiding in monoidal categories] A braiding in a monoidal category $\mathcal{C}$ consists of a natural family of isomorphisms

$$c = \{c_{V,W} : V \otimes W \to W \otimes V\}, \tag{2.4.5}$$

where V, W run over all abjects of $\mathcal{C}$, such that for any three objects U, V, W, we have

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W), \tag{2.4.6}$$

$$c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}). \tag{2.4.7}$$

Definition 2.4.3. [Twist in monoidal categories] A twist in a monoidal category $\mathcal{C}$ with a braiding c consists of a natural family of isomorphisms

$$\theta = \{\theta_V : V \to V\}, \tag{2.4.8}$$

where V runs over all objects of $\mathcal{C}$, such that for any two objects V, W of $\mathcal{C}$, we have

$$\theta_{V\otimes W} = c_{W,V} c_{V,W} (\theta_V \otimes \theta_W). \tag{2.4.9}$$

Let $\mathcal{C}$ be a monoidal category. Assume that to each object V of $\mathcal{C}$ there are associated an object V^* of $\mathcal{C}$ and two morphisms

$$b_V : \mathbf{1} \to V \otimes V^*,\ d_V : V^* \otimes V \to \mathbf{1}. \tag{2.4.10}$$

Definition 2.4.4. [Duality in monoidal categories] The rule $V \to (V^*, b_V, d_V)$ is called a *duality* in $\mathcal{C}$ if the following identities are satisfied:

$$(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V, \tag{2.4.11}$$

$$(d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}. \tag{2.4.12}$$

Note that we do not require that $(V^*)^* = V$.

We say that the duality in $\mathcal{C}$ is *compatible* with the braiding c and the twist θ in $\mathcal{C}$ if for any object V of $\mathcal{C}$, we have

$$(\theta_V \otimes \mathrm{id}_{V^*}) b_V = (\mathrm{id}_V \otimes \theta_{V^*}) b_V. \tag{2.4.13}$$

Definition 2.4.5. [Ribbon categories] A ribbon category is a monoidal category $\mathcal{C}$ equipped with a braiding c, a twist θ, and a compatible duality $(*, b, d)$. A ribbon category is called strict if its underlying monoidal category is strict.

Let $\mathcal{C}$ be a ribbon category. Denote $K = K_{\mathcal{C}}$ the semigroup $\mathrm{End}(\mathbf{1})$ with the multiplication induced by the composition of morphisms and the unit element $\mathrm{id}_{\mathbf{1}}$.

Definition 2.4.6. [Traces and dimensions] For an endomorphism $f : V \to V$ of an object V, we define its *trace* $\mathrm{tr}(f) \in K$ to be the following composition:

$$\mathrm{tr}(f) = d_V c_{V,V^*}((\theta_V f) \otimes \mathrm{id}_{V^*}) b_V : \mathbf{1} \to \mathbf{1}. \tag{2.4.14}$$

For an object V in $\mathcal{C}$, we define its *dimension* $\dim(V)$ by the formula

$$\dim(V) = \mathrm{tr}(\mathrm{id}_V) = d_V c_{V,V^*}(\theta_V \otimes \mathrm{id}_{V^*}) b_V \in K. \tag{2.4.15}$$

2.4.2 Definition of Modular Tensor Categories

Definition 2.4.7. [Ab-categories] A category $\mathcal{C}$ is said to be an Ab-category if for any pair of its objects V, W, the set $\mathrm{Hom}(V, W)$ of $\mathcal{C}$-morphisms $V \to W$ is an additive abelian group and the composition of morphisms is bilinear.

Let $\mathcal{C}$ be an monoidal Ab-category. The composition of morphisms, considered as multiplication in $\mathrm{End}(\mathbf{1}) = \mathrm{Hom}(\mathbf{1}, \mathbf{1})$, renders this abelian group a ring with unit $\mathrm{id}_{\mathbf{1}}$. This ring is commutative. It is called the ground ring of $\mathcal{C}$ and denoted by $K_{\mathcal{C}}$ or by K.

Combining the definition of Ab-category with the definitions of section 2.4.1 we come to the notion of a ribbon Ab-category. This is a monoidal Ab-category equipped with braiding, twist, and compatible duality.

Let $\mathcal{C}$ be a ribbon Ab-category. For any $k \in K$ and any object V of $\mathcal{C}$, the morphism $k \otimes \mathrm{id}_V : V \to V$ is called multiplication by k in V.

Definition 2.4.8. [Simple objects] An object V of $\mathcal{C}$ is said to be simple if the formula $k \mapsto k \otimes \mathrm{id}_V$ defines a bijection $K \to \mathrm{End}(V)$.

For example, the unit object $\mathbf{1}$ is simple.

Here is a convenient characterization of simple objects: an object V of $\mathcal{C}$ is simple if and only if $\mathrm{End}(V)$ is a free K-module of rank 1. Indeed, if V is simple then $\mathrm{End}(V) \simeq K$ with the generator id_V. Conversely, if $\mathrm{End}(V) \simeq K$ with a free generator x then $\mathrm{id}_V = kx$ and $x^2 = k'x$ with $k, k' \in K$ Hence $x = \mathrm{id}_V x = kx^2 = kk'x$. Therefore k is invertible in K and id_V is a free generator of $\mathrm{End}(V)$.

Let $\{V_i\}_{i\in I}$ be a family of objects of a ribbon Ab-category $\mathcal{C}$.

Definition 2.4.9. [Domination] An object V of $\mathcal{C}$ is dominated by the family $\{V_i\}_{i\in I}$ if there exist a finite set $\{V_{i(r)}\}_r$ of objects of this family (possibly with repetitions which means that the same object may appear several times) and a family of morphisms $\{f_r : V_{i(r)} \to V,\ g_r : V \to V_{i(r)}\}_r$ such that

$$\mathrm{id}_V = \sum_r f_r g_r. \tag{2.4.16}$$

Here $i(r) \in I$ for all r.

The definition of domination may be reformulated as follows: V dominated by $\{V_i\}_{i\in I}$ if the images of the pairings

$$\{(g, f) \mapsto fg : \mathrm{Hom}(V, V_i) \otimes_K \mathrm{Hom}(V_i, V) \to \mathrm{End}(V)\}_{i\in I}$$

additively generate $\mathrm{End}(V)$.

For $i, j \in I$, set

$$\dim(i) = \dim(V_i) \in K \text{ and } S_{i,j} = \mathrm{tr}(c_{V_j,V_i} \circ c_{V_i,V_j}) \in K$$

where K is the ground ring of $\mathcal{C}$. Note that $S_{i,j} = S_{j,i}$. Thus, $S = [S_{i,j}]_{i,j\in I}$ is a symmetric square matrix over K and

$$S_{0,i} = S_{i,0} = \mathrm{tr}(\mathrm{id}_{V_i}) = \dim(i).$$

Definition 2.4.10. [Modular tensor categories] A modular category is a pair consisting of a ribbon Ab-category $\mathcal{C}$ and a finite family $\{V_i\}_{i\in I}$ of simple objects of $\mathcal{C}$ satisfying the following four axioms.

1. (*Normalization axiom*) There exists $0 \in I$ such that $V_0 = \mathbf{1}$.
2. (*Duality axiom*) For any $i \in I$, there exists $i^* \in I$ such that the object V_{i^*} is isomorphic to $(V_i)^*$.
3. (*Axiom of domination*) All objects of $\mathcal{C}$ are dominated by the family $\{V_i\}_{i\in I}$.
4. (*Non-degeneracy axiom*) The square matrix $S = [S_{i,j}]_{i,j\in I}$ is invertible over K.

Remarks:(cf. [Row06]) In a semisimple ribbon Ab-category $\mathcal{C}$ with finitely many simple classes the set of simple classes generates a semiring over K under $\otimes$ and $\oplus$. This ring is called the *Grothendieck semiring* and denoted by $Gr(\mathcal{C})$. If $\{V_0 = \mathbf{1}, V_1, ..., V_{n-1}\}$ is the set of representatives of the simple objects in $\mathcal{C}$, the *rank* of $\mathcal{C}$ is n. We have

$$V_i \otimes V_j \cong \sum_k N_{i,j}^k V_k \tag{2.4.17}$$

for some $N_{i,j}^k \in \mathbb{N}$. These structure coefficients of $Gr(\mathcal{C})$ are called the *fusion coefficients* of $\mathcal{C}$ and (2.4.17) is sometimes called a fusion rule. If we fixed the order of the simple objects as above, the fusion coefficients give us a representation of $Gr(\mathcal{C})$ via $V_i \to N_i$ where $N_i = (N_i)_{k,j} = (N_{i,j}^k)$ is called the fusion matrix associated to V_i. If i^* is the index of the simple object V_i^*, the braiding and associativity constraints give us:

$$N_{i,j}^k = N_{j,i}^k = N_{i,k*}^{j*} = N_{i*,j*}^{k*},\ N_{i,j}^0 = \delta_{i,j*}.$$

The first column (and row) of the matrix S consists of the categorical dimensions of the simple objects, i.e., $S_{i,0} = \dim(V_i)$. We denote these dimensions by d_i. We also have that $S_{i,j} = S_{j,i} = S_{i*,j*}$. Since the twist $\theta_V \in \mathrm{End}(V)$ for any object V, θ_V is a scalar map. We denote this scalar by θ_i. And we get a useful formula

$$S_{i,j} = \frac{1}{\theta_j \theta_j} \sum_k N_{i*,j}^k d_k \theta_k. \tag{2.4.18}$$

Provided $\mathcal{C}$ is modular the matrix S determines the fusion rules via the *Verlinde formula*

$$N_{i,j}^k = \sum_t \frac{S_{i,t} S_{j,t} S_{k*,t}}{D^2 S_{0,t}} \tag{2.4.19}$$

where $D^2 = \sum_i d_i^2$. This formula corresponds to the following fact: the columns of the matrix S are simultaneous eigenvectors for the fusion matrices N_i, and the categorical dimensions are eigenvalues.

If we set $T = (\delta_{i,j}\theta_i)_{ij}$ then the map:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to S, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to T$$

defines a projective representation of the *modular group* $SL_2(\mathbb{Z})$. In fact, by renormalizing S and T one gets the representation of $SL_2(\mathbb{Z})$.

Chapter 3

Genera of Codes, Lattices, and VOAs

3.1 Code genus

Recall That $C^{\perp} = \{y \in \mathbb{F}_q^n : x \cdot y = 0 \text{ for all } x \in C\}$ is a dual code of a code C. For a doubly even binary code C, we know that $C \subset C^{\perp}$. For a code word c in C, we denote $w(c)$ the weight of c.

We define the *weight signature* of $C^{\perp}$ to be the set

$$W = \{w(c) : w(c) \bmod 4 \text{ , } c \text{ is a codeword in } C^{\perp}\}.$$

Definition 3.1.1. Two doubly even codes with the same lengths are said to be in the same genus if and only if they have the same dimensions and their dual codes have the same weight signatures.

Note that there are three possible genera for the doubly even codes of a given length n and dimension k: odd genus with $\{\overline{0}, \overline{1}, \overline{3}\}$, even genus with $\{\overline{0}, \overline{2}\}$, and even genus with $\{\overline{0}\}$.

3.2 Lattice genus

Let L be an integral lattice and the vector space $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ with the induced symmetric bilinear form. By Sylvester's law of inertia, this form can be diagonalized, i.e., there exists a basis $\{e_1, ...e_n\}$ of V such that the inner product of vectors $v = \sum_i v_i e_i$ and $w = \sum_i w_i e_i$ is given by $\sum_i \epsilon_i v_i w_i$ where $\epsilon_i \in \{-1, 0, 1\}$. The multiplicity of $-1, 0$, and 1 among the ϵ_is is invariant under the choice of diagonalizing basis.

We can say that the lattice L has a *signature* (l_+, l_-) where l_+ is the number of positive ϵ_i and l_- is the number of negative ϵ_i.

Let L be an even lattice in $\mathbb{R}^n$. The quotient group $A := L^*/L$ is a finite abelian group. We define the mapping

$$q_A : A \to \mathbb{Q}/2\mathbb{Z}, \text{ by } q_A(x + L) = x^2 + 2\mathbb{Z}, \text{ where } x \in L^*.$$

Then q_A is called a quadratic form on A, i.e., the discriminant form of L^*/L.

Definition 3.2.1. [Nik79] (cf. [Höh03]) Two even lattices belong to the same genus if and only if their signatures and discriminant forms are the same.

Note that the number of isometry classes contained in the genus is called the class number.

3.3 VOA genus

Recall that $C_2(V)$ is the subspace of V spanned by all elements of the form $A_{-2} \cdot B$ for all $A, B \in V$

Theorem 3.3.1. *(cf. theorem 4.6 in [Hua08]) Let V be a simple vertex operator algebra. Assume that*

1. *$V_n = 0$ for $n < 0$, $V_0 = \mathbb{C}\mathbf{1}$ and V' is isomorphic to V as a V-module.*

2. *Every $\mathbb{N}$-graded weak V-module is completely reducible.*

3. *V satisfy the C_2 cofiniteness condition.*

Then the category of V-modules has a natural structure of modular tensor category.

The VOAs we consider here are assumed to be unitary an satisfy the conditions in above theorem.

Definition 3.3.2. [Höh03] Two VOAs are said to be in the same genus if and only if their associated modular tensor categories (MTCs) and central charges are the same. We denote the genus by $\mathcal{G}(\mathcal{C}, c)$, where $\mathcal{C}$ is the corresponding MTC and c is the central charge of the VOAs.

Note that the MTC $\mathcal{C}$ determines the central charge c only modulo 8.

Then we have the following commutative diagram:

$$\begin{array}{ccccc} \text{Doubly even binary codes} & \overset{L_C}{\hookrightarrow} & \text{Even lattices} & \overset{V_L}{\hookrightarrow} & \text{VOAs} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Code genus} & \hookrightarrow & \text{Lattice genus} & \hookrightarrow & \text{VOA genus} \end{array}$$

L_C is the lattice constructed from a code C and V_L is the VOA associated with the lattice L_C.

Note that $C^\perp/C \simeq L^*/L = A$ and $q_A = w/4 \pmod{\mathbb{Q}/2\mathbb{Z}}$ and (A, q_A) defines the MTC of V_L.

Chapter 4

Codes Type Genera of Lattices

4.1 Constructing Lattices from Binary Codes

From binary codes we can construct lattices. Take the standard lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ and consider the reduction mod 2:

$$\rho : \mathbb{Z}^n \to (\mathbb{Z}/2\mathbb{Z})^n = \mathbb{F}_2^n.$$

This is a group homomorphism. Let C be an $[n, k, d]$-code. Since $\mathbb{F}_2^n/C \cong \mathbb{F}_2^{n-k}$, C is a subgroup of index

$$|\mathbb{F}_2^n/C| = 2^{n-k}$$

of $\mathbb{F}_2^n$. Therefore $\rho^{-1}(C)$, the preimage of C in $\mathbb{Z}^n$, is a subgroup of index 2^{n-k} of $\mathbb{Z}^n$. In particular $\rho^{-1}(C)$ is a free abelian group of rank n. Therefore $\rho^{-1}(C)$ is a lattice in $\mathbb{R}^n$. One has

$$\mathrm{vol}(\mathrm{\mathbb{R}^n}/\rho^{-1}(\mathrm{C})) = |\mathrm{\mathbb{Z}^n}/\rho^{-1}(\mathrm{C})|\mathrm{vol}(\mathrm{\mathbb{R}^n}/\mathrm{\mathbb{Z}^n}) = 2^{\mathrm{n-k}}.$$

Definition 4.1.1. We denote L_C a lattice constructed from a binary code C and $L_C := \dfrac{1}{\sqrt{2}}\rho^{-1}(C)$.

The set L_C is a lattice in $\mathbb{R}^n$. Let $x, y \in L_C$. Then x and y can be written

$$x = \frac{1}{\sqrt{2}}(c + 2z), y = \frac{1}{\sqrt{2}}(c' + 2z')$$

for some $c, c' \in \{0,1\}^n$ representing codewords in C and some $z, z' \in \mathbb{Z}^n$. By abuse of notation we shall identify in the sequel $\mathbb{F}_2^n$ with the subset $\{0,1\}^n$ of $\mathbb{Z}^n$ and write briefly $c, c' \in C$.

Proposition 4.1.2 (cf. [Ebe02]). *Let C be a linear code.*
(i) $C \subset C^\perp$ if and only if L_C is an integral lattice.
(ii) C is doubly even if and only if L_C is an even lattice.
(iii) C is self-dual if and only if L_C is unimodular.

4.2 Classification of the Lattice Genera

We begin with the structure of the genera of lattices which are constructed from codes. Here we establish the following result.

Theorem 4.2.1. *[Genera of lattice arising from codes] Let C be a doubly even code of length n and dimension k. If n is not divisible by 4, then the genus of C depends just on n and k. If n is divisible by 4, then there are two possible genera for fixed n and k depending on whether the dual code $C^\perp$ of C contains vectors of odd weight or not.*

Proof: The discriminant group of L_C^*/L_C can be identified with $C^\perp/C$ which is isomorphic to the abelian group $(\mathbb{Z}/2\mathbb{Z})^{n-2k}$ and so the genus of L_C depends on n (the rank of L_C) and k by the above characterization theorem for genera (cf. Section 3.2 and [Höh03]). The quadratic form q_A for L_C is given by $q_A(x) = w(x)/4$ (depending on the normalization of q_A) for x a codeword in a coset of $C^\perp/C$. It remains to show that if n is not divisible by 4 only one genus can occur and if n is divisible by 4 two cases are possible. If n is odd, then the dual code $C^{\perp}$ of a code C of length n always contains a vector of weight n, i.e. an odd weight. So there is only one genus which is of odd type. If n is even and not divisible by 4, then any codeword of a code C of length n has at least two coordinates 0s. So there exists a codeword of C that has the last two coordinates 0s and its dual code contains a vector (codeword) such that all coordinates are 0s except the last coordinate 1. So this vector has an odd weight, and hence this gives a genus of odd type. If n is divisible by 4, then there are two possible cases here. First, C contains a vector of weight n. In this case, $C^{\perp}$ contains only even-weight vectors, and hence we have a genus of even type. Otherwise, C contains no vector of weight n. Then there exists a codeword which has last four coordinates 0s, and hence its dual code contains the vector which all coordinates 0s except the last coordinate 1. So the dual code contains at least one odd-weight vector, and therefore this gives a genus of odd type. **q.e.d.**

We denote a genus of codes C of type $[n, k]$ by $\mathcal{G}(n, k, t)$ where t is the type of the genus which is *odd* or *even* depending on whether the dual code $C^{\perp}$ of C contains vectors of odd weight or not. Note that for n not divisible by 4 the type is always odd since $C^{\perp}$ always contains odd weight vectors by the argument in the proof of the above theorem.

Lemma 4.2.2 (Which genera actually can occur from codes)**.** *For $k = 0$ and $n \equiv 0 \pmod 4$, the only realized genus is odd. Depending on $n \pmod 4$ and $n \pmod 8$ the maximal k for which a code exists is given in the following table:*

$n(mod4)/n(mod8)$	$0/0(o)$	$0/0(e)$	$0/4(o)$	$0/4(e)$	$1/1$	$1/5$	$2/2$	$2/6$	$3/3$	$3/7$
k	$\frac{n}{2}-1$	$\frac{n}{2}$	$\frac{n}{2}-2$	$\frac{n}{2}-1$	$[\frac{n}{2}]$	$[\frac{n}{2}]-1$	$\frac{n}{2}-1$	$\frac{n}{2}-1$	$[\frac{n}{2}]-1$	$[\frac{n}{2}]$

where $[\frac{n}{2}]$ means the largest integer that is less than or equal to $\frac{n}{2}$, (o) for odd genus and (e) for even genus.

Proof: Note that the maximal possible k is $\dfrac{n}{2}$ since $C \subset C^{\perp}$ and $\dim C^{\perp} = n-k$. For $n < 4$, k $= 0$. For $n \equiv 0 \pmod 4$ and $k = 0$, $C^{\perp}$ always contains vectors of odd weight.

We will prove this lemma by dividing the proof into 3 cases depending on $n \pmod 4$

and $n \pmod 8$.

Case 1 $n \equiv 0 \pmod 4$ and $n \equiv 4 \pmod 8$. $n = 4, 12, 20, ...$

In this case the code C of length n contains vectors which either have all coordinates 1s (called type A) or have a multiple of 4 of coordinates 0s (called type B). If $n = 4$, then there are only two vectors ; (1,1,1,1)-type A and (0,0,0,0)-type B. So for the odd case, the dimension is 0 and for the even case, the dimension is 1.

For an *odd case*, C contains only vectors of type B. Without loss of generality, suppose that each vector in C has the last four components 0s. Then $C^{\perp}$ contains at least four more linearly independent vectors including $(0, ..., 0, 1, 0, 0, 0), (0, ..., 0, 0, 1, 0, 0), (0, ..., 0, 0, 1, 0)$, and $(0, ..., 0, 0, 0, 0, 1)$. Hence dim $C^{\perp}$ is at least $k + 4$, i.e. dim C is at most $k = \frac{n}{2} - 2$.

For an *even case*, C contains a vector of type A. Without loss of generality, suppose that the rest of the vectors in C have the last four components 0s. Since $C^{\perp}$ contains only even weight vectors, $C^{\perp}$ must contain the vectors $(0, ..., 0, 1, 1, 0, 0)$ and $(0, ..., 0, 0, 0, 1, 1)$. Thus dim $C^{\perp}$ is at least $k + 2$, i.e. dim C is at most $k = \frac{n}{2} - 1$.

Case 2 $n \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 8$. $n = 8, 16, ...$

Note that for $n > 4$ we have $n = m + 4$ where $m \equiv 0 \pmod 4$ and $m \equiv 4 \pmod 8$. A code C of length n can be constructed by adding four more coordinates to each vector in the code C_m of length m. So $C \cong C_m \oplus (x, x, x, x)$ (using $\oplus$ here means we expand the vector of length m to be of length n by adding four more coordinates at the end of each vector in C_m). Then $c_1 = c_1^m \oplus (0, 0, 0, 0), c_2 = c_2^m \oplus (0, 0, 0, 0), ..., c_{k_m} = c_{k_m}^m \oplus (0, 0, 0, 0)$, where $c_1^m, ..., c_{k_m}^m$ are basis vectors in C_m, and hence $c_1, ..., c_{k_m}$ are k_m basis vectors in C.

We know that any doubly even binary code C_m of length m is always a subset of its dual code $C_m^{\perp}$ and hence dim $C_m \leq$ dim $C_m^{\perp}$. Then when we add four more coordinates to each vectors in C_m there are at least one more basis vectors for a code C other than those we get from the vectors in C_m. Then these vectors are also contained in $C^{\perp}$.

For an *odd case*, we can construct a code C from a code C_m which can be either odd or even. If C_m is of odd type, then there are at least four 0s in each vector and we have four more coordinates added. Hence we will have at most three more basis vectors in C(other than the basis vectors from C_m). Thus dim C = dim $C_m + 3 = \frac{m}{2} - 2 + 3 = \frac{m}{2} + 1 = \frac{n-4}{2} + 1 = \frac{n}{2} - 2 + 1 = \frac{n}{2} - 1$. If C_m is of even type, then there are only two more basis vectors can be added to make C contains no vector of type A. Hence dim C = dim $C_m + 2$ $= \frac{m}{2} - 1 + 2 = \frac{m}{2} + 1 = \frac{n-4}{2} + 1 = \frac{n}{2} - 2 + 1 = \frac{n}{2} - 1$. Thus $k = \frac{n}{2} - 1$.

Similarly for an even case, since C can contains a vector of type A , we will get four more basis vectors if C_m is of odd type and three more basis vectors if C_m is of even type. So dim $C = \frac{n}{2}$.

Case 3 $n \equiv l \pmod 4$ where $l = 1, 2,$ or 3. $n = 5, 6, 7, 9, 10, 11, 13, ...$

Note that we only have odd cases here. Then $n - l \equiv 0 \pmod 4$.

Subcase 3.1 $n - l \equiv 4 \pmod 8$

We can construct a code C of length n by adding l more coordinates to each vector in the code C' of length $n - l$. For a code C' of odd type, since each basis vector in C' is of type B, i.e. it has at least four coordinates 0s. After adding l more coordinates to each of its vectors, there is at most one more basis vector to be added in C. For a code C' of even type, we will have zero, two and two more basis vectors in C after adding one, two , and three coordinates to each vector in C' respectively. So we will have k as the following;

For $l = 1$, dim C = dim $C' = \frac{n-1}{2} - 1 = \frac{n-1}{2} - 1 = [\frac{n}{2}] - 1$.

For $l = 2$, dim C = dim $C' + 2 = \frac{n-2}{2} - 1 + 2 = \frac{n-2}{2} = \frac{n}{2} - 1$.

For $l = 3$, dim C = dim $C' + 2 = \frac{n-3}{2} - 1 + 2 = \frac{n-1}{2} = [\frac{n}{2}]$.

Subcase 3.2 $n - l \equiv 0 \pmod 8$

Similarly, to have vector in C of type B after adding l coordinates to each vector in C', we will have no more basis vector in C, if C' if of even type and we will have one more basis vector in C if C' is of odd type. So we will have k as the following;

For $l = 1$, dim C = dim $C' = \frac{n-1}{2} = \frac{n-1}{2} = [\frac{n}{2}]$.

For $l = 2$, dim C = dim $C' = \frac{n-2}{2} = \frac{n-2}{2} = \frac{n}{2} - 1$.

For $l = 3$, dim C = dim $C' = \frac{n-3}{2} = \frac{n-1}{2} - 1 = [\frac{n}{2}] - 1$.

Table 4.1: *The maximal dimension k for n from 4 to 16*

n	$4(o)$	$4(e)$	5	6	7	$8(o)$	$8(e)$	9	10	11	$12(o)$	$12(e)$	13	14	15	$16(o)$	$16(e)$
k	0	1	1	2	3	3	4	4	4	4	4	5	5	6	7	7	8

q.e.d.

Definition 4.2.3. A lattice genus is called *code type* genus if every lattice L in this genus is of the form L_C for some code C. Otherwise we call it *non code type* genus.

Our aim in this section is to classify all code type genera. Clearly, only the genera as in Proposition 4.2.7 below can be code type and have to investigated further.

We start with some easy observations.

Lemma 4.2.4. *If $\mathcal{G}(n,k,t)$ is a non code type genus then $\mathcal{G}(n+8,k+4,t)$ is also non code type.*

Proof: Since $\mathcal{G}(n,k,t)$ is a non code type, there exist a lattice L in the genus which is not realizable. We claim that the lattice $L \oplus E_8$ which belongs to the genus $\mathcal{G}(n+8,k+4,t)$ is also not realizable. Indeed, if otherwise $L \oplus E_8 \cong K_C$ for a code C of type $[n+8,k+4]$ then there exists a code D of type $[n,k]$ such that $C \cong D \oplus e_8$, where e_8 is a code of type $[8,4]$. Since e_8 is self-dual, $L_{e_8} = E_8$ is unimodular, i.e. disc $(E_8) = 1$. Then $K_C \cong L_D \oplus E_8$ and thus $L \cong L_D$, a contradiction. **q.e.d.**

The Lemma shows that if we have found an n_0 such that all $\mathcal{G}(n,k,t)$ for $n = n_0, \ldots,$ $n_0 + 7$ are non code type so are all $\mathcal{G}(n,k,t)$ for $n > n_0$ and we have reduced the problem to check of finitely many cases.

Lemma 4.2.5. *If $\mathcal{G}(n,k,t)$ is a non code type genus then $\mathcal{G}(n,l,t)$ for $l < k$ is also non code type.*

Lemma 4.2.6. *Every genus $\mathcal{G}(n,k,t)$ for $n \geq 32$ is non code type.*

Proof: By computation, $\mathcal{G}(n,k,t)$ for $n = 17$ to $n = 23$ and for any k are non code type. The genera of the even unimodular lattices of dimension 24 are non code type (cf. [Ebe02]). Then by Lemma 4.2.2, Lemma 4.2.4, and Lemma 4.2.5, $\mathcal{G}(n,k,t)$ for $n \geq 32$ is non code type. **q.e.d.**

Proposition 4.2.7. *The following genera are code type:*

n	$\mathcal{G}(n,k,t)$
1	$\mathcal{G}(1,0,\text{odd})$
2	$\mathcal{G}(2,0,\text{odd})$
3	$\mathcal{G}(3,0,\text{odd})$
4	$\mathcal{G}(4,0,\text{odd})$, $\mathcal{G}(4,1,\text{even})$
5	$\mathcal{G}(5,0,\text{odd})$, $\mathcal{G}(5,1,\text{odd})$
6	$\mathcal{G}(6,0,\text{odd})$, $\mathcal{G}(6,1,\text{odd})$, $\mathcal{G}(6,2,\text{odd})$
7	$\mathcal{G}(7,0,\text{odd})$, $\mathcal{G}(7,1,\text{odd})$, $\mathcal{G}(7,2,\text{odd})$, $\mathcal{G}(7,3,\text{odd})$
8	$\mathcal{G}(8,0,\text{odd})$, $\mathcal{G}(8,1,\text{odd})$, $\mathcal{G}(8,1,\text{even})$, $\mathcal{G}(8,2,\text{odd})$, $\mathcal{G}(8,2,\text{even})$,$\mathcal{G}(8,3,\text{odd})$, $\mathcal{G}(8,3,\text{even})$, $\mathcal{G}(8,4,\text{even})$
9	$\mathcal{G}(9,1,\text{odd})$, $\mathcal{G}(9,2,\text{odd})$, $\mathcal{G}(9,3,\text{odd})$, $\mathcal{G}(9,4,\text{odd})$
10	$\mathcal{G}(10,2,\text{odd})$, $\mathcal{G}(10,3,\text{odd})$, $\mathcal{G}(10,4,\text{odd})$
11	$\mathcal{G}(11,3,\text{odd})$, $\mathcal{G}(11,4,\text{odd})$
12	$\mathcal{G}(12,4,\text{odd})$, $\mathcal{G}(12,4,\text{even})$, $\mathcal{G}(12,5,\text{odd})$
13	$\mathcal{G}(13,5,\text{odd})$
14	$\mathcal{G}(14,6,\text{odd})$
15	$\mathcal{G}(15,7,\text{odd})$
16	$\mathcal{G}(16,8,\text{even})$

The complete information of the code type genera is in Table 4.5 below.

Proof: In order to prove this proposition, we have to compare the number of the permutation equivalence classes of doubly even binary codes with the number of lattices in the corresponding genus. To do this, we use the data base from http://www.rlmiller.org/de_codes and it is also stated in [DFG+11] as in Table 4.2 below. Then we compute the number of lattices in each genus corresponding to each code by using the computer algebra **Magma**. First, we construct each code C of length n and dimension k and then we construct the corresponding lattice L_C from the code C. Next, we use the command **Genus(L);** to construct the genus $\mathcal{G}(n, k, t)$. Finally, we can find the number of lattices in each genus and then we can compare these numbers with the numbers of codes. But when n is larger than 17, Magma cannot compute these numbers. So we need to find each lattice in each genus directly. To do this, we have to find the lattices in the largest dimension only (in some cases the two largest dimensions are computed) and we stop when we find a large enough number of lattices in the genus to show that it is a non code type genus. The results are in Table 4.3 below. By comparing the numbers in Tables 4.2 and 4.3 and applying Lemma 4.2.5, we get the result as in Table 4.4. The complete information about this computation is explained in Appendix A and the Magma source codes in the computation is in Appendix C. **q.e.d.**

Note that Magma cannot compute the exact number of lattices in some genera that have n too large. In this case, we use another method to find different lattices in the genera for the largest k for each n. So we only find the least possible number of lattices in those particular genera which proved that the numbers of lattices in those genera are more than the number of the corresponding codes and apply Lemma 4.2.5 to get the result in Table 4.4.

Remark: For the genera corresponding to the codes of length 24, the genus $\mathcal{G}(24, 12, \text{even})$ consists of the even unimodular lattices of dimension 24. By Corollary 3.7 in [Ebe02], there are 24 such lattices and there are only 9 of them coming from the doubly even self-dual codes. Hence this genus is non code type.

Theorem 4.2.8. *The genera listed in Proposition 4.2.7 are the only code type genera.*

Proof: By computation with Magma and Lemma 4.2.2, Lemma 4.2.4, Lemma 4.2.5, Lemma 4.2.6 and the remark above, now that all $\mathcal{G}(n, k, t)$ for $n \geq 17$ are non code type genera. So the code type genera are listed in Proposition 4.2.7. **q.e.d.**

Table 4.2: *The number of distinct permutation classes of doubly even (n,k) codes*

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
1	1											
2	1											
3	1											
4(o)	1											
4(e)	0	1										
5	1	1										
6	1	1	1									
7	1	1	1	1								
8(o)	1	1	1	1	0							
8(e)	0	1	1	1	1							
9	1	2	2	2	1							
10	1	2	3	3	2							
11	1	2	3	4	3							
12(o)	1	2	4	5	5	0						
12(e)	0	1	1	2	2	2						
13	1	3	5	8	8	4						
14	1	3	7	12	14	9	4					
15	1	3	7	15	20	15	8	2				
16(o)[1]	1	4	10	23	38	36	23	4	0			
16(e)[1]	0							5	2			
17	1	4	10	25	45	50	34	14	3			
18	1	4	13	34	72	94	79	35	9			
19	1	4	13	40	94	146	141	75	19			
20(o)[1]	1	5	17	57	158	295	353	231	84	0		
20(e)[1]	0									10		
21	1	5	17	63	194	439	629	494	198	38		
22	1	5	21	83	298	812	1481	1465	740	187	25	
23	1	5	21	95	387	1287	2970	3811	2362	714	119	11
n/k	0	1	2	3	4	5	6	7	8	9	10	11

[1] known number of codes but cannot classify the exact numbers of either odd or even type codes.

Table 4.3: *The number of distinct lattices in the genus $\mathcal{G}(n, k, t)$*

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
1	1											
2	1											
3	1											
4(o)	1	0										
4(e)	0	1										
5	1	1										
6	1	1	1									
7	1	1	1	1								
8(o)	1	1	1	1	0							
8(e)	0	1	1	1	1							
9	2	2	2	2	1							
10	2	3	3	3	2							
11	2	3	4	4	3							
12(o)	3	4	6	6	5	0						
12(e)	0	2	2	3	2	2						
13	-	-	-	-	9	4						
14	-	-	-	-	18	10	4					
15	-	-	-	-	-	-	9	2				
16(o)	6	-	-	-	-	-	-	5	0			
16(e)	0							6	2			
17	-	-	-	-	-	-	-	-	4			
18	-	-	-	-	-	-	-	-	10[2]			
19	-	-	-	-	-	-	-	-	20[2]			
20(o)	-	-	-	-	-	-	-	-	67[3]	0		
20(e)										12[2]		
21	-	-	-	-	-	-	-	-	-	39[2]		
22	-	-	-	-	-	-	-	-	-	-	27[2]	
23	-	-	-	-	-	-	-	-	-	-	-	12[2]
n/k	0	1	2	3	4	5	6	7	8	9	10	11

[2] There are at least the indicated number of lattices in the genus.

[3] There are 84 codes of type (20,8) but we do not know the exact numbers of codes of either odd or even types.By computation,we found 53 odd type codes and 19 even type codes and there are at least 67 lattices in the odd type genus. After adding this number with the number of even type codes, we have the total of 86 which is exceed the number of the code (20,8). So the odd genus is non code type.

Table 4.4: *Types of distinct lattice genus $\mathcal{G}(n, k, t)$, C: code type lattice genus, NC: non code type lattice genus by computation, NC*: non code type lattice by Lemma 4.2.4, X: no lattice genus*

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
1	C											
2	C											
3	C											
4(o)	C											
4(e)	X	C										
5	C	C										
6	C	C	C									
7	C	C	C	C								
8(o)	C	C	C	C	X							
8(e)	X	C	C	C	C							
9	NC	C	C	C	C							
10	NC	NC	C	C	C							
11	NC	NC	NC	C	C							
12(o)	NC	NC	NC	NC	C	X						
12(e)	X	NC	NC	NC	C	C						
13	NC*	NC*	NC*	NC*	NC	C						
14	NC*	NC*	NC*	NC*	NC*	NC	C					
15	NC	NC	NC*	NC*	NC*	NC*	NC	C				
16(o)	NC	NC*	NC*	NC*	NC*	NC*	NC*	NC	X			
16(e)	X	NC*	NC*	NC*	NC*	NC*	NC*	NC	C			
17	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC			
18	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC			
19	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC			
20(o)	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC	X		
20(e)	X	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC		
21	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC		
22	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC	
23	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC*	NC
n/k	0	1	2	3	4	5	6	7	8	9	10	11

The following table consists of the complete information of the code type lattice genera.

Table 4.5: Code type lattice genera $\mathcal{G}(n,k,t)$

n	k	t	Codes(C)	Lattices (L_C)	$\lvert L^*/L\rvert$
1	0	odd	t_1	A_1	1
2	0	odd	t_2	$2A_1$	2
3	0	odd	t_3	$3A_1$	3
4	0	odd	t_4	$4A_1$	4
4	1	even	d_4	D_4	2
5	0	odd	t_5	$5A_1$	5
5	1	odd	$t_1 \oplus d_4$	$A_1 \oplus D_4$	3
6	0	odd	t_6	$6A_1$	6
6	1	odd	$t_2 \oplus d_4$	$2A_1 \oplus D_4$	4
6	2	odd	d_6	D_6	2
7	0	odd	t_7	$7A_1$	7
7	1	odd	$t_3 \oplus d_4$	$3A_1 \oplus D_4$	5
7	2	odd	$t_1 \oplus d_6$	$A_1 \oplus D_6$	3
7	3	odd	e_7	E_7	1
8	0	odd	t_8	$8A_1$	8
8	1	odd	$t_4 \oplus d_4$	$4A_1 \oplus D_4$	6
8	1	even	h_8 (or $d_4 * d_4$)*	$L(h_8)$ (or $D_4 * D_4$)	6
8	2	odd	$t_2 \oplus d_6$	$2A_1 \oplus D_6$	4
8	2	even	$d_4 \oplus d_4$	$2D_4$	4
8	3	odd	$t_1 \oplus e_7$	$A_1 \oplus E_7$	2
8	3	even	d_8	D_8	2

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Table 4.5 – *Continued from previous page*

n	k	t	Codes(C)	Lattices (L_C)	$\|L^*/L\|$
8	4	even	e_8	E_8	0
9	1	odd	$t_5 \oplus d_4$	$5A_1 \oplus D_4$	7
			$t_1 \oplus h_8$	$A_1 \oplus L(h_8)$	
9	2	odd	$t_3 \oplus d_6$	$3A_1 \oplus D_6$	5
			$t_1 \oplus d_4 \oplus d_4$	$A_1 \oplus 2D_4$	
9	3	odd	$t_2 \oplus e_7$	$2A_1 \oplus E_7$	3
			$t_1 \oplus d_8$	$A_1 \oplus D_8$	
9	4	odd	$t_1 \oplus e_8$	$A_1 \oplus E_8$	1
10	2	odd	$t_4 \oplus d_6$	$4A_1 \oplus D_6$	6
			$t_2 \oplus d_4 \oplus d_4$	$2A_1 \oplus 2D_4$	
			$C_{102} = \begin{bmatrix} 1111000000 \\ 0011111111 \end{bmatrix}$	$L(C_{102})$	
10	3	odd	$t_3 \oplus e_7$	$3A_1 \oplus E_7$	4
			$d_4 \oplus d_6$	$D_4 \oplus D_6$	
			$t_2 \oplus d_8$	$2A_1 \oplus D_8$	
10	4	odd	$t_2 \oplus e_8$	$2A_1 \oplus E_8$	2
			d_{10}	D_{10}	
11	3	odd	$t_4 \oplus e_7$	$4A_1 \oplus E_7$	5
			$t_3 \oplus d_8$	$3A_1 \oplus D_8$	
			$t_1 \oplus d_4 \oplus d_6$	$A_1 \oplus D_4 \oplus D_6$	
			$C_{113} = \begin{bmatrix} 11111111000 \\ 00111111110 \\ 10100000101 \end{bmatrix}$	$L(C_{113})$	
11	4	odd	$t_3 \oplus e_8$	$3A_1 \oplus E_8$	3

Continued on next page

Table 4.5 – *Continued from previous page*

n	k	t	Codes(C)	Lattices (L_C)	$\|L^*/L\|$
			$d_4 \oplus e_7$	$D_4 \oplus E_7$	
			$t_1 \oplus d_{10}$	$A_1 \oplus D_{10}$	
12	4	odd	$t_4 \oplus e_8$	$4A_1 \oplus E_8$	4
			$d_6 \oplus d_6$	$2D_6$	
			$t_2 \oplus d_{10}$	$2A_1 \oplus D_{10}$	
			$t_1 \oplus d_4 \oplus e_7$	$A_1 \oplus D_4 \oplus E_7$	
			$C_{124} = \begin{bmatrix} 100011101111 \\ 010011010000 \\ 001010111111 \\ 000101110000 \end{bmatrix}$	$L(C_{124})$	
12	4	even	$d_4 \oplus d_8$	$D_4 \oplus D_8$	4
			$C_{124e1} = \begin{bmatrix} 100011111110 \\ 010011000001 \\ 001010111111 \\ 000101111111 \end{bmatrix}$	$L(C_{124e1})$	
12	5	even	$d_4 \oplus e_8$	$D_4 \oplus E_8$	2
			d_{12}	D_{12}	
13	5	odd	$t_1 \oplus d_4 \oplus e_8$	$A_1 \oplus D_4 \oplus E_8$	5
			$d_6 \oplus e_7$	$D_6 \oplus E_7$	
			$t_1 \oplus d_{12}$	$A_1 \oplus D_{12}$	

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Table 4.5 – *Continued from previous page*

n	k	t	Codes(C)	Lattices (L_C)	$\lvert L^*/L\rvert$
			$C_{135} = \begin{bmatrix} 1000011110111 \\ 0100010001001 \\ 0010011111110 \\ 0001000001101 \\ 0000100001011 \end{bmatrix}$	$L(C_{135})$	
14	6	odd	$d_6 \oplus e_8$	$D_6 \oplus E_8$	2
			$e_7 \oplus e_7$	$2E_7$	
			d_{14}	D_{14}	
			$C_{146} = \begin{bmatrix} 10000011111110 \\ 01000010000101 \\ 00100011111011 \\ 00010001000101 \\ 00001000100101 \\ 00000100000111 \end{bmatrix}$	$L(C_{146})$	
15	7	odd	$e_7 \oplus e_8$	$E_7 \oplus E_8$	1
			e_{15}	D_{14}^+	
16	8	even	$e_8 \oplus e_8$	$2E_8$	0
			e_{16}	D_{16}^+	

* The code h_8 is a subcode of $d_4 \oplus d_4$ generated by the codeword of length 8. We can construct the code h_8 by gluing two of the code d_4 together. And the result will be $d_4 * d_4$ (the symbol here represents the gluing of the two codewords) in the notation of the basic code of type d_4. So the corresponding lattice is $D_4 * D_4$.

Chapter 5

Genera of Vertex Operator Algebras arising from the Small Modular Tensor Categories

In this chapter we will classify the genera of vertex operator algebras (VOAs) arising from small modular tensor categories (MTCs). By small modular tensor categories we mean the MTCs of rank less than or equal to 4. Note that the VOAs that we consider here have to satisfy the conditions in the Theorem 3.3.1 and they are assumed to be unitary.

Recall that the VOA has a finite number of simple modules $V = M^1,\ M^2, \ldots M^r$. Each has a q-graded character

$$\mathrm{ch}\ M^j = \mathrm{Tr}_{M^j} q^{L_0^{M^j} - c/24} = \sum_n \dim M_n^j q^{n-c/24} = q^{h_j - c/24} \sum_n \dim M_{n+h_j}^j q^n, \tag{5.0.1}$$

where M_n^j is the subspace of M^j on which $L_0^{M^j}$ acts by multiplication by n, c is the central charge of V, h_j is the *conformal weight* of M^j, and $q = e^{2\pi i\tau}$. And these VOA modules have the structure of the modular tensor categories. So we can classify the genera of the VOAs using their associated MTCs. Proposition 3.1 in [DM04] states that for each state $u \in V$

which is homogeneous of weight k with respect to the operator L_0, the r-tuple $Z(u,\tau) = (Z_{M^1}(u,\tau), ..., Z_{M^r}(u,\tau))$ is a *vector-valued modular form* of weight k with respect to the representation ρ. Note that $Z_{M^j}(u,\tau) = \mathrm{Tr}_{M^j} o(u) q^{L_0^{M^j} - c/24} = q^{h_j - c/24} \sum_n \dim M^j_{n+h_j} o(u) q^n$, where $o(u)$ is the zero mode of the homogeneous components of u (see detail in [DM04]).

For any VOA V, recall that $\mathcal{G}(\mathcal{C}(V), c)$ is the genus of V, where $\mathcal{C}(V)$ is the MTC associated with the VOA V and c is the central charge of V.

The family $\{\mathrm{ch}\ M^i\}_{i=1,..n}$ is a vector valued modular function of a representation ρ of $SL_2(\mathbb{Z})$ determined by $\mathcal{C}(V)$. Note that $h_i \pmod 1$ is given by $\mathcal{C}(V)$ and for a unitary VOA V, $h_i \geq 0$ and $c \geq 0$.

Let $M(\rho, c)$ be the space consisting of vector valued modular forms for the representation ρ with pole orders at most $c/24$ at infinity. Then ch M^i is an element of $M(\rho, c)$ and $M(\rho, c)$ depends only on the genus of V. Our objective is to describe the space of vector valued modular forms $M(\rho, c)$.

We apply the idea of the fundamental matrix of the representation of the modular group in [BG07] to determine the spaces $M(\rho, c)$ arising from the genera of the VOAs. As a result, the first column of a fundamental matrix consists of characters of the corresponding MTC, i.e., characters of a VOA V and its modules. Moreover, the first entry of a fundamental matrix contains a dimension of some Lie algebras in its second term. Then we apply this fact to classify the possible Kac-Moody subVOAs $\tilde{V}_1$ and then we can classify the genera of the VOAs arising from each MTC.

5.1 Small MTCs

The following table consists of the list of MTCs of rank 1, 2, 3, and 4 which we call small MTCs. We use the classification of the MTCs from [RSW09] which also gives all the S-matrices of the MTCs (see Table B.1).

Table 5.1: The small MTCs

No.	$\mathcal{C}$	n	c (mod 8)	h_i	No.	$\mathcal{C}$	n	c (mod 8)	h_i
1	t_m	1	0	0	18	qs_4	4	1	0, 1/8, 1/8, 1/2
2	qs_2	2	1	0, 1/4	19	$\overline{qs_4}$	4	7	0, 7/8, 7/8, 1/2
3	$\overline{qs_2}$	2	7	0, 3/4	20	qn_4	4	5	0, 1/2, 5/8, 5/8
4	$Lee-Yang$	2	14/5	0, 2/5	21	$\overline{qn_4}$	4	3	0, 3/8, 3/8, 1/2
5	$\overline{Lee-Yang}$	2	26/5	0, 3/5	22	qu_2	4	8	0, 0, 0, 1/2
6	qs_3	3	2	0, 1/3, 1/3	23	qv_2	4	4	0, 1/2, 1/2, 1/2
7	$\overline{qs_3}$	3	6	0, 2/3, 2/3	24	$qs_2 \otimes qs_2$	4	2	0, 1/4, 1/4, 1/2
8	$Ising1$	3	1/2	0, 1/2, 1/16	25	$\overline{qs_2} \otimes \overline{qs_2}$	4	6	0, 3/4, 3/4, 1/2
9	$\overline{Ising1}$	3	15/2	0, 1/2, 15/16	26	$qs_2 \otimes \overline{qs_2}$	4	8	0, 3/4, 1/4, 1
10	$Ising2$	3	3/2	0, 1/2, 3/16	27	$qs_2 \otimes LY$	4	19/5	0, 2/5, 1/4, 13/20
11	$\overline{Ising2}$	3	13/2	0, 1/2, 13/16	28	$\overline{qs_2} \otimes LY$	4	49/5	0, 2/5, 3/4, 3/20
12	$Ising3$	3	5/2	0, 1/2, 5/16	29	$qs_2 \otimes \overline{LY}$	4	31/5	0, 3/5, 1/4, 17/20
13	$\overline{Ising3}$	3	11/2	0, 1/2, 11/16	30	$\overline{qs_2} \otimes \overline{LY}$	4	61/5	0, 3/5, 3/4, 7/20
14	$Ising4$	3	7/2	0, 1/2, 7/16	31	$LY \otimes LY$	4	28/5	0, 2/5, 2/5, 4/5
15	$\overline{Ising4}$	3	9/2	0, 1/2, 9/16	32	$LY \otimes \overline{LY}$	4	8	0, 3/5, 2/5, 1
16	$3fieldsx$	3	8/7	0, 2/7, 6/7	33	$\overline{LY} \otimes \overline{LY}$	4	52/5	0, 3/5, 3/5, 1/5
17	$\overline{3fieldsx}$	3	48/7	0, 5/7, 1/7	34	$4fieldsx$	4	10/3	0, 2/3, 2/9, 1/3
					35	$\overline{4fieldsx}$	4	14/3	0, 1/3, 7/9, 2/3

From the table, column 2 ($\mathcal{C}$) consists of the names of the MTCs which we follow the notation from the database [Dat]. The rank and the central charge (mod 8) of each MTC is shown in column 3 (n) and 4 (c) respectively. And the last column (h_i) consists of the conformal weights of each MTC.

Recall that there exists a representation $\rho : SL_2(\mathbb{Z}) \longleftrightarrow GL_n(\mathbb{C})$ of the modular group $SL_2(\mathbb{Z})$ sending its generating elements, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, to the matrices S and T of a MTC.

The representation ρ could be either irreducible or reducible. For an reducible representation ρ, we have to decompose it into a direct sum $\rho = \rho_1 \oplus \cdots \oplus \rho_s$ of its irreducible components ρ_i for our method.

We use Magma to decompose the representation ρ into a direct sum of its irreducible components (see Appendix C for the source codes), and we also get the corresponding canonical basis vectors for each irreducible representation ρ_i. Note that the idea of the decomposition is also mentioned in the appendix in [BG07]. The result of the decomposition of the representation of each MTC is in Table B.1. Column 6 in Table B.1 describes the decomposition into irreducible components and the number $\underline{m}$ represents a dimension of each component.

5.2 Characters of MTCs

5.2.1 Scalar and Vector Valued Modular Forms

In this section we give details about scalar valued modular form (see any text book of the related title or Section 2.2 in [Ebe02]) and vector valued modular form.

The group

$$SL_2(\mathbb{Z}) = \{ \ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1\}$$

acts on the complex upper half-plane $\mathbb{H}$ by fractional linear transformations

$$\tau \mapsto g(\tau) = \frac{a\tau + b}{c\tau + d}.$$

The center $\{\pm 1\}$ of $SL_2(\mathbb{Z})$ acts trivially. The quotient $G := SL_2(\mathbb{Z})/\{\pm 1\}$ is called the *modular group*.

Let S and T be the elements of G represented by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The elements S and T act on $\mathbb{H}$ as follows:

$$S : \tau \mapsto -\frac{1}{\tau},\ T : \tau \mapsto \tau + 1.$$

G is generated by these elements.

Let k be an even positive integer. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *modular form of weight* k, if the following conditions are satisfied :

(i) $f(\frac{a\tau+b}{c\tau+d}) = (c\tau + d)^k f(\tau)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

(ii) f has a power series expansion in $q = e^{2\pi i\tau}$, i.e., f is holomorphic at infinity $\tau = i\infty$.

Next we will define a vector-valued modular form (c.f. [KM+04]) as follows:

Let $\rho : \Gamma \to GL_d(\mathbb{C})$ denote a d-dimensional representation of $\Gamma = SL_2(\mathbb{Z})$, $k \in \mathbb{R}$ an arbitrary real number. A function

$$F(\tau) = \begin{pmatrix} f_1(\tau) \\ \vdots \\ f_d(\tau) \end{pmatrix}, \text{where } \tau \in \mathbb{H}$$

from the complex upper half-plane $\mathbb{H}$ to $\mathbb{C}^d$ is a *vector-valued modular form* of weight k if the following conditions are satisfied:

1. For all $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$F(\tau) \mid_k V(\tau) = \rho(V)F(\tau)$$

2. Each component function $F_j(\tau)$ has a convergent q-expansion meromorphic at infinity

$$F_j(\tau) = \sum_{n>h_j} a_n(j) q^{\frac{n}{N_j}}$$

with N_j a positive integer, h_j an integer (maybe negative) and $q = \exp(2\pi i \tau)$.

The operator $|_k V$ is defined by

$$F |_k V(\tau) = F |_k^v V(\tau) = v(V^{-1})(c\tau + d)^{-k} F(V_\tau)$$

with a multiplier system v with respect to Γ.

5.2.2 The Fundamental Matrix

We define the fundamental matrix of the representation as in [BG07].

Consider a matrix representation $\rho : \Gamma \to GL_d(\mathbb{C})$ whose kernel contains $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and for which $T = \rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a diagonal matrix of finite order. We associate to ρ the set $\mathcal{M}(\rho)$ of all those maps $\mathbb{X} : \mathbb{H} \to \mathbb{C}^d$ which are holomorphic in the upper half plane $\mathbb{H}$, transform according to ρ, that is

$$\mathbb{X}\left(\frac{a\tau + b}{c\tau + d}\right) = \rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbb{X}(\tau) \qquad (5.2.1)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$, and have only finite order poles at the cusps. So $\mathbb{X}$ is a vector-valued modular form. Since $\rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is diagonal of finite order, there exists a diagonal matrix Λ (the *exponent matrix*) such that

$$\rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \exp(2\pi i\Lambda), \tag{5.2.2}$$

the diagonal elements of Λ being rational numbers.

The space $\mathcal{M}(\rho)$ is an infinite-dimensional linear space over $\mathbb{C}$, a basis being provided by the maps $\mathbb{X}^{(\xi,n)} \in \mathcal{M}(\rho)$ which have a pole of order $n > 0$ at the ξth position. We call these $\mathbb{X}^{(\xi,n)}$ the *canonical basis vectors*.

Let

$$J(\tau) = q^{-1} + \sum_{n=1}^{\infty} c(n)q^n = q^{-1} + 196884q + ... \tag{5.2.3}$$

denote the Hauptmodul of $SL_2(\mathbb{Z})$, i.e., the generator of the field of modular functions for $SL_2(\mathbb{Z})$. Multiplication by J takes the space $\mathcal{M}(\rho)$ to itself, in other words $\mathcal{M}(\rho)$ is a $\mathbb{C}[J]$-module of finite rank and the canonical basis vectors satisfy the *recursion relations*

$$\mathbb{X}^{(\xi;m+1)} = J(\tau)\mathbb{X}^{(\xi,m)} - \sum_{n=1}^{m-1} c(n)\mathbb{X}^{(\xi,m-n)} - \sum_{\eta} \mathcal{X}_{\eta}^{(\xi;m)}\mathbb{X}^{(\eta;1)}, \tag{5.2.4}$$

where

$$\mathcal{X}_{\eta}^{(\xi;m)} = \mathbb{X}^{(\xi;m)}[0]_{\eta} = \lim_{q\to 0} \left([q^{-\Lambda}\mathbb{X}^{(\xi;m)}(q)]_{\eta} - q^{-m}\delta_{\xi\eta}\right) \tag{5.2.5}$$

denotes the “constant part” of $\mathbb{X}^{(\xi;m)}$. These recursion relations allow us to express each canonical basis vector $\mathbb{X}^{(\xi;m)}$ in terms of the $\mathbb{X}^{(\xi,1)}$s. Note that the $\mathbb{X}^{(\xi,1)}$ are linearly independent over the field $\mathbb{C}(J)$ of modular functions, and thus the $\mathbb{C}[J]$-module $\mathcal{M}(\rho)$ has rank d.

There is a second set of relations, “the *differential relations*”, between the canonical basis vectors. They follow from the fact that the differential operator

$$\nabla = \frac{\mathcal{E}(\tau)}{2\pi i}\frac{d}{d\tau} \tag{5.2.6}$$

maps $\mathcal{M}(\rho)$ to itself, where

$$\mathcal{E}(\tau) = \frac{E_{10}(\tau)}{\Delta(\tau)} = \sum_{n=-1}^{\infty} \mathcal{E}_n q^n = q^{-1} - 240 - 141444q - ... \tag{5.2.7}$$

is the quotient of the Eisenstein series of weight 10 by the discriminant form $\Delta(\tau) = q\Pi_{n=1}^{\infty}(1-q^n)^{24}$ of weight 12. The action of ∇ on the canonical basis vectors gives the differential relations

$$\nabla \mathbb{X}^{(\xi;m)} = (\Lambda_{\xi\xi} - m)\sum_{n=-1}^{m-1} \mathcal{E}_n \mathbb{X}^{(\xi;m-n)} + \sum_{\eta} \Lambda_{\eta\eta}\mathcal{X}_{\eta}^{(\xi;m)}\mathbb{X}^{(\eta;1)}. \tag{5.2.8}$$

The compatibility of the recursion and differential relations requires that

$$\nabla \mathbb{X}^{(\xi;1)} = (J-240)(\Lambda_{\xi\xi}-1)\mathbb{X}^{(\xi;1)} + \sum_{\eta}(1+\Lambda_{\eta\eta}-\Lambda_{\xi\xi})\mathcal{X}_{\eta}^{(\xi;1)}\mathbb{X}^{(\eta;1)}, \tag{5.2.9}$$

which is a first-order ordinary differential equation - the *compatibility equation* - for the $\mathbb{X}^{(\xi;1)}$s.

From equation (5.2.9), we define the *fundamental matrix* as follow

$$\Xi(\tau)_{\xi\eta} = [\mathbb{X}^{(\eta;1)}(\tau)]_{\xi}, \tag{5.2.10}$$

whose columns span over $\mathbb{C}[J]$ the module $\mathcal{M}(\rho)$. Then equation (5.2.9) takes the form

$$\frac{1}{2\pi i}\frac{d\Xi(\tau)}{d\tau} = \Xi(\tau)\mathfrak{D}(\tau), \tag{5.2.11}$$

where

$$\mathfrak{D}(\tau) = \frac{1}{\mathcal{E}(\tau)}\{(J(\tau) - 240)(\Lambda - 1) + \mathcal{X} + [\Lambda, \mathcal{X}]\}, \tag{5.2.12}$$

$\mathcal{X}_{\xi\eta} = \mathcal{X}_{\xi}^{(\eta;1)}$ is the *characteristic matrix* and $[\mathcal{X}, \Lambda] = \mathcal{X}\Lambda - \Lambda\mathcal{X}$.

Taking the boundary condition

$$q^{1-\Lambda_{\xi\xi}}\Xi(q)_{\xi\eta} = \delta_{\xi\eta} + O(q) \text{ as } q \to 0, \tag{5.2.13}$$

one can solve equation (5.2.11), provided one knows the exponent matrix Λ and the characteristic matrix $\mathcal{X}$, determining then from equation (5.2.4) the canonical basis vectors $\mathbb{X}^{(\xi;m)}$.

Note that the exponent matrix has to satisfy the following condition:

$$\mathrm{Tr}(\Lambda) = \frac{5d}{12} + \frac{1}{4}\mathrm{Tr}(S) + \frac{2}{3\sqrt{3}}\mathrm{Re}(e^{-\pi i/6}\mathrm{Tr}(U)) \tag{5.2.14}$$

where d is the dimension of ρ and we use the notations

$$S = \rho\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } U = \rho\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

.

The structure of the $\mathbb{C}[J]$-module $\mathcal{M}(\rho)$ is completely determined by the fundamental matrix $\Xi(\tau)$ (cf. [BG07]), once an exponent matrix Λ has been chosen. The fundamental matrix is itself completely determined by the pair $(\Lambda, \mathcal{X})$ of exponent and characteristic matrices, namely as the solution of the compatibility equation (5.2.11) satisfying the boundary condition equation (5.2.13). We consider the pair $(\Lambda, \mathcal{X})$ as the basis data characterizing the representation ρ.

Remark : The fundamental matrix Ξ allows us to determine the space $M(\rho, c) \subset \mathcal{M}(\rho)$ for $c < 24$.

5.2.3 Method of finding the Fundamental Matrix

We use the method in [BG07] to find the fundamental matrices corresponding to the conformal weights of the MTCs.

Consider the function

$$\mathfrak{j}(\tau) = \frac{984 - J(\tau)}{1728}, \tag{5.2.15}$$

which maps the upper half-plane $\mathbb{H}$ onto the complex plane $\mathbb{C}$. It is modular invariant and satisfies the differential equation

$$\nabla \mathfrak{j} = 1728\mathfrak{j}(\mathfrak{j} - 1). \tag{5.2.16}$$

Let us consider the fundamental matrix as a function of $\mathfrak{j}$. Then, by applying the chain rule and equation (5.2.16), one gets the following form of the compatibility equation;

$$\frac{d\Xi(\mathfrak{j})}{d\mathfrak{j}} = \Xi\left(\frac{\mathcal{A}}{2\mathfrak{j}} + \frac{\mathcal{B}}{3(\mathfrak{j}-1)}\right), \tag{5.2.17}$$

with

$$\mathcal{A} = \frac{31}{36}(1 - \Lambda) - \frac{1}{864}(\mathcal{X} + [\Lambda, \mathcal{X}]), \tag{5.2.18}$$

$$\mathcal{B} = \frac{41}{24}(1 - \Lambda) + \frac{1}{576}(\mathcal{X} + [\Lambda, \mathcal{X}]). \tag{5.2.19}$$

As a function of $\mathfrak{j}$ the fundamental matrix is not single valued - its multivaluedness, i.e., the monodromy (that is the behavior of an object as it winding around a singularity) of equation (5.2.17), is described by the representation ρ. In particular, the monodromies around $\mathfrak{j}=0$, $\mathfrak{j}=1$, $\mathfrak{j}=\infty$ are given by

$$S=\rho\begin{pmatrix}0&-1\\1&0\end{pmatrix},U=\rho\begin{pmatrix}0&-1\\1&-1\end{pmatrix},T=\rho\begin{pmatrix}1&1\\0&1\end{pmatrix}$$

respectively. Because the residues of equation (5.2.17) at these points are $\mathcal{A}/2, \mathcal{B}/3$ and $\Lambda-1$, the matrices S and U are conjugate to $\exp(\pi i\mathcal{A})$ and $\exp(\pi i\mathcal{B}/3)$, respectively, and one has $SU=T^{-1}=\exp(-2\pi i\Lambda)$. We find that the monodromy group of the abstract hypergeometric equation (5.2.17) is precisely the image of ρ.

There are some restrictions for the matrices $\mathcal{A}$ and $\mathcal{B}$ as follow:

Spectral condition: The possible eigenvalues of $\mathcal{A}$ are 0 or 1, while those of $\mathcal{B}$ are either 0, 1 or 2.

In particular, this condition implies that the characteristic polynomials of $\mathcal{A}$ and $\mathcal{B}$ read

$$\det(z-\mathcal{A})=z^{d-\alpha}(z-1)^{\alpha}, \tag{5.2.20}$$

$$\det(z-\mathcal{B})=z^{d-\beta_1-\beta_2}(z-1)^{\beta_1}(z-2)^{\beta_2}, \tag{5.2.21}$$

where d denotes their dimensions, while the multiplicities α, β_1 and β_2 are given by

$$\alpha=\mathrm{Tr}(\mathcal{A}), \tag{5.2.22}$$

$$\beta_1=2\mathrm{Tr}(\mathcal{B})-\mathrm{Tr}(\mathcal{B}^2), \tag{5.2.23}$$

$$\beta_2=\frac{1}{2}(\mathrm{Tr}(\mathcal{B}^2)-\mathrm{Tr}(\mathcal{B})). \tag{5.2.24}$$

The quadruple $(d, \alpha, \beta_1, \beta_2)$ of non-negative integers is a very important discrete invariant of the representation ρ, which we will call its *signature*.

It follows from equations (5.2.20) and (5.2.21) that the minimal polynomials of $\mathcal{A}$ and $\mathcal{B}$ divide $z(z-1)$, respectively, $z(z-1)(z-2)$. Since any matrix is a root of its minimal polynomial, the spectral condition may be expressed as

$$\mathcal{A}(\mathcal{A}-1) = \mathcal{B}(\mathcal{B}-1)(\mathcal{B}-2) = 0. \tag{5.2.25}$$

Of the four matrices $\Lambda, \mathcal{X}, \mathcal{A}$ and $\mathcal{B}$, any two determine the other two, e.g., equations (5.2.18) and (5.2.19) imply that $\mathcal{B} = 3(1 - \Lambda - \mathcal{A}/2)$. Inserting this expression into equation (5.2.11), one gets the following system of algebraic equations:

$$\begin{aligned} &\mathcal{A}^2 = \mathcal{A}, \\ &\mathcal{A}\Lambda\mathcal{A} = -\frac{17}{18}\mathcal{A} - 2(\mathcal{A}\Lambda^2 + \Lambda\mathcal{A}\Lambda + \Lambda^2\mathcal{A}) + 3(\mathcal{A}\Lambda + \Lambda\mathcal{A}) - 4\Lambda^3 + 8\Lambda^2 - \frac{44}{9}\Lambda + \frac{8}{9} \end{aligned} \tag{5.2.26}$$

That is , for a given exponent matrix Λ, the matrix $\mathcal{A}$ has to satisfy equations (5.2.26). Once a solution to equations (5.2.26) is known, the corresponding characteristic matrix $\mathcal{X}$ may be determined from equation (5.2.18).

To find the fundamental matrix $\Xi(\tau)$,we do as the following:

- Begin from a given exponent matrix Λ and then solve the equations (5.2.26) to get the matrix $\mathcal{A}$.
- Use the matrix $\mathcal{A}$ to find the characteristic matrix $\mathcal{X}$ by solving equation (5.2.18).
- Use the exponent matrix Λ and the characteristic matrix $\mathcal{X}$ to get the fundamental matrix $\Xi(\tau)$ by solving equation (5.2.11).

Recall that the matrices S and T of any MTC correspond to some representation of the modular group $SL_2(\mathbb{Z})$. In particular, they are the images of the representation ρ of the generators of the modular group. Note that from [Hua08], the twist $\theta_V : V \to V$ is given by the operator $e^{2\pi i L_0}$.

For a given MTC $\mathcal{C}$ of rank n with the central charge c and conformal weights $h_1, h_2, ..h_n$. We define $\lambda_i = h_i - \dfrac{c}{24}$ and set

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & \ddots & & \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}.$$

Then Λ is the exponent matrix corresponding to the MTC $\mathcal{C}$.

Note that we may have to modify some of the λ_is to be λ_i mod 1 in order that Λ satisfies equation (5.2.14).

Follow the method above, one gets the corresponding fundamental matrix Ξ of the representation of the MTC $\mathcal{C}$. In this thesis, we explore only the small MTCs up to $n = 4$.

5.2.4 Results

Now we will give some examples of how to compute the fundamental matrices and the corresponding characters of the given MTCs.

Example 5.2.1. The fundamental matrix of the representation corresponding to the VOA genus $\mathcal{G}(\mathbf{qs_2}, \mathbf{1})$.

The MTC qs_2 is of rank 2 with central charge 1 and conformal weights 0, 1/4. We have $\lambda_1 = 0 - 1/24 = -1/24$ and $\lambda_2 = 1/4 - 1/24 = 5/24$ and the exponent matrix is

$$\Lambda = \begin{pmatrix} \frac{23}{24} & 0 \\ 0 & \frac{5}{24} \end{pmatrix}.$$

Note that Λ has to satisfy equation (5.2.14) so we have to modify λ_1 to λ_1 mod 1.

Next we solve the equations (5.2.26) to get the matrix $\mathcal{A}$ (we use Mathematica in this computation and the source codes are explained in the Appendix C). We get

$$\mathcal{A} = \begin{pmatrix} \frac{7}{216} & f(1,2) \\ \frac{1463}{46656 f(1,2)} & \frac{209}{216} \end{pmatrix}.$$

Note that $f(1,2)$ is a parameter since there are infinitely many possible solutions for the equations (5.2.26).

Next we solve the equation (5.2.18) to get the characteristic matrix with the parameter;

$$\mathcal{X} = \begin{pmatrix} 3 & -\frac{3456}{7} f(1,2) \\ -\frac{2926}{27 f(1,2)} & -247 \end{pmatrix}.$$

Finally, we solve the equation (5.2.11) and get the fundamental matrix with the parameter as follow

$$\Xi = q^{\Lambda} \begin{pmatrix} q^{-1} + 3 + 4q + 7q^2 + \cdots & -\frac{3456}{7} f(1,2) - \frac{2464128}{77} f(1,2) q - \cdots \\ -\frac{2926}{27 f(1,2)} - \frac{2926 q}{27 f(1,2)} - \cdots & q^{-1} - 247 - 86241 q - \cdots \end{pmatrix}.$$

To find the value of the parameter $f(1,2)$, we compare the first column of the fundamental matrix with the known characters of the Wess-Zumino-Novikov-Witten (WZW) model of level 1 based on the corresponding Lie algebra (cf. [BG07]). In this case, qs_2 corresponds to the affine Kac-Moody Lie algebra $A_{1,1}$ (the WZW model A_1 level 1). By comparing the first column of Ξ with the corresponding characters of $A_{1,1}$ we get the following result.

The characteristic matrix is

$$\mathcal{X} = \begin{pmatrix} 3 & 26752 \\ 2 & -247 \end{pmatrix}$$

and the fundamental matrix is

$$\Xi = q^{\Lambda}\begin{pmatrix} q^{-1}+3+4q+7q^2+\cdots & 26752+1734016q+46091264q^2+\cdots \\ \\ 2+2q+6q^2+\cdots & q^{-1}-247-86241q-4182736q^2-\cdots \end{pmatrix}.$$

Note that the first entry in the characteristic matrix represents the dimension of the corresponding Lie algebra and this number also appears in the second term of the first entry of the fundamental matrix.

For the MTCs of rank larger than 2, there are more than one parameter in the resulting matrix $\mathcal{A}$. But we can also compare the first column of the fundamental matrix with the characters of the corresponding known affine Kac-Moody Lie algebra(WZW model) to get the values of the parameters.

The representation ρ of qs_2 is irreducible so there is only one component. The matrices S and T of qs_2 with central charge 1 are

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \text{ and } T = \begin{pmatrix} e^{\frac{23i\pi}{12}} & 0 \\ 0 & e^{\frac{5i\pi}{12}} \end{pmatrix}.$$

The canonical basis vectors are $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and $v_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Note that these vectors are the canonical basis vectors corresponding to $\mathbb{X}^{(\xi,1)}$s which determine the order and position of the pole of the representation (see Section 5.2.2).

Recall that the entries in the first column of the fundamental matrix are the characters of the VOA V and its modules, i.e., the characters of the MTC qs_2. So we have the following theorem.

Theorem 5.2.2. *The characters of the VOAs and their modules in $\mathcal{G}(qs_2, 1)$ have the following forms:*

$$\text{ch } M^1 = q^{23/24}\left(q^{-1}+3+4q+7q^2+13q^3+\cdots\right)$$

$$\text{ch } M^2 = q^{5/24}\left(2+2q+6q^2+8q^3+\cdots\right).$$

Example 5.2.3. The characters of the MTC corresponding to the VOA genus $\mathcal{G}(\mathbf{qs_2} \otimes \mathbf{qs_2}, \mathbf{10})$.

The MTC $qs_2 \otimes qs_2$ is of rank 4 with conformal weights $h_1 = 0, h_2 = 1/4, h_3 = 1/4$, and $h_4 = 1/2$.

We have $h'_1 = -5/12$, $h'_2 = -1/6$, $h'_3 = -1/6$, and $h'_4 = 1/12$.

After decomposing the representation ρ of the MTC $qs_2 \otimes qs_2$, we have $\rho = \rho_1 \oplus \rho_2$, where ρ_1 is a one dimensional irreducible representation with h'_2 forming its exponent matrix Λ_1 and ρ_2 is a three dimensional irreducible representation with h'_1, h'_3, and h'_4 forming its exponent matrix Λ_2. The S_i and T_i matrices are

$$S_1 = (-1) \text{ and } S_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \text{ and}$$

$$T_1 = (e^{\frac{5i\pi}{3}}) \text{ and } T_2 = \begin{pmatrix} e^{\frac{7i\pi}{6}} & 0 & 0 \\ 0 & e^{\frac{5i\pi}{3}} & 0 \\ 0 & 0 & e^{\frac{i\pi}{6}} \end{pmatrix}$$

The canonical basis vectors are $v_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$, and $v_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$.

The linear combination of the v_js is

$$av_1 + bv_2 + dv_3 + ev_4. \tag{5.2.27}$$

Since the canonical basis vectors determine the order and the position of the pole of the representation, the resulting basis vectors determine the coefficients of the direct sum of the vectors (columns) in the fundamental matrix.

Note that v_2 is the canonical basis vector for ρ_1. v_1, v_3, and v_4 are the canonical basis vectors for ρ_2.

We have the following matrices of the representations ρ_1 and ρ_2:

- The exponent matrices

  $\Lambda_1 = (\text{-}1/6)$

  $\Lambda_2 = \mathrm{Diag}(\frac{7}{12}, \frac{5}{6}, \frac{1}{12})$

- The characteristic matrices

  $\mathcal{X}_1 = (4)$

  $$\mathcal{X}_2 = \begin{pmatrix} 190 & 32 & 4928 \\ 512 & -4 & -22528 \\ 20 & -8 & 66 \end{pmatrix}$$

- The fundamental matrices

$$\Xi_1 = q^{\Lambda_1} \begin{pmatrix} q^{-1} + 4 - 196870q - 43775016q^2 - 2767606261q^3 - ... \end{pmatrix}$$

$$\Xi_2 = q^{\Lambda_2} \begin{pmatrix} q^{-1} + 190 + 5245q + ... & 32 + 192q + 800q^2 + ... & 4928 + 896896q + ... \\ 512 + 10240q + ... & q^{-1} - 4 + 6q - ... & -22528 - 2547712q - ... \\ 20 + 1160q + ... & -8 - 80q - 408q^2 - ... & q^{-1} + 66 + 86647q + ... \end{pmatrix}.$$

Recall that a character of a VOA module is of the form

$$\mathrm{ch}\ M^j = q^{h_j - c/24} \sum_{n \geq 0} \dim M^j_{n+h_j} q^n = q^{h'_j} \sum_{n \geq 0} \dim M^j_{n+h_j} q^n.$$

So $h'_j = h_j - c/24$ determines whether ch M^j (as a vector valued modular form) has a pole or not, i.e., if $h'_j < 0$, then ch M^j has a pole at infinity.

Remarks :

- ch M^1 always has a pole at infinity since $h_1 = 0$. The coefficient of the first term of ch M^1 has to be 1 since it is the dimension of the subspace $V_0 \simeq \mathbb{C}\mathbf{1}$.

- If $h'_j < 0$, $j \neq 0$, then the corresponding basis vector v_j contributes to the pole of ch M^j. So there is a combination of the first column of the fundamental matrix and the other columns which correspond to $h'_j (< 0)$s, i.e., the columns generated by v_js. So the coefficient of v_j in equation (5.2.27) has to be nonnegative.

In this case, $h_3' = -1/6$ with the basis vector v_3 contributes to the pole of ch M^2. Since v_3 generates the second column of the fundamental matrix of ρ_2, there is a combination of the first and second columns of the fundamental matrix of ρ_2.

Note that the entry of the fundamental matrix of ρ_1 (which corresponds to h_2') does not contribute to any pole. Since $5/6 = h_2' \pmod 1 \neq h_1' \pmod 1 = 7/12$. So this entry cannot contribute to the pole of ch M^1. It also cannot contribute to the pole of ch M^2, since it will give a pole of order larger than one. That is the q-expansion in the fundamental matrix of ρ_1 is $q^{5/6}\left(q^{-2} + 4q^{-1} - 196870 - 43775016q - 2767606261q^2 - \cdots \right)$. So the values of the coefficients in equation (5.2.27) are $a = 1$, $b = 0$, $d \geq 0$, and $e = 0$. Hence the characters of a VOA and its modules in this genus is the combination of the first and second columns in the fundamental matrix of ρ_2 and we have the following theorem.

Theorem 5.2.4. *The characters of the VOAs and their modules in $\mathcal{G}(qs_2 \otimes qs_2, 10)$ have the following forms:*

$$\text{ch } M^1 = q^{7/12}(q^{-1} + (190 + 32d) + (5245 + 192d)q + (62150 + 800d)q^2 + \cdots)$$

$$\text{ch } M^2 = q^{5/6}(dq^{-1} + (512 - 4d) + (10240 + 6d)q + (107520 - 8d)q^2 + \cdots)$$

$$\text{ch } M^3 = q^{5/6}(dq^{-1} + (512 - 4d) + (10240 + 6d)q + (107520 - 8d)q^2 + \cdots)$$

$$\text{ch } M^4 = q^{1/12}((20 - 8d) + (1160 - 80d)q + (19324 - 408d)q^2 + \cdots)$$

where d is a suitable nonnegative integer.

Remarks :

- d has to be a nonnegative integer since it appears as the dimension of the submodule in ch M^i.

- ch M^1 contains the dimension of the corresponding reductive Lie algebra V_1 as the second term. So $190 + 32d$ is the dimension of a reductive Lie algebra V_1.

- V_1 generates an affine Kac-Moody subVOA $\tilde{V}_1$.

Theorem 5.2.5. *Table B.2 consists of the exponent matrices* (Λ) *and the characteristic matrices* ($\mathcal{X}$) *of the representations of the MTCs corresponding to the VOA genera* $\mathcal{G}(\mathcal{C}, c)$ *and Table B.3 consists of the characters of the MTCs corresponding to the VOA genera* $\mathcal{G}(\mathcal{C}, c)$.

Table B.2 contains the exponent matrix Λ_j and the characteristic matrix $\mathcal{X}_j$ of each irreducible component which contributes to the pole of the character ch M^i. Note that we omit the exponent matrix and the characteristic matrix of the representation which does not contribute to the pole of ch M^i.

Remark We use Table 5.3 in [Höh07] as the reference for some of the characters of the affine Kac-Moody Lie algebras. In some cases, there is no explicit reference for the characters of the corresponding Lie algebras but we can do as the following:

1. Compute the fundamental matrix Ξ_1 of a MTC with central charge c.
2. Then take a tensor product of Ξ_1 and the fundamental matrix corresponding to $E_{8,1} \otimes E_{8,1}$ (the fundamental matrix of the trivial MTC with central charge 16).
3. Compute the fundamental matrix Ξ_2 of the same MTC as in step 1 but with central charge $c + 16$.
4. Compare the first column of the resulting matrix from step 2 with the first column of the fundamental matrix Ξ_2.

5.3 Genera of VOAs arising from small MTCs with central charge at most 16

We use the characters of the MTCs that we get from the computation in Section 5.2 to classify the genera of the VOAs arising from the small MTCs. The component V_1 of the VOA V has a structure of a reductive Lie algebra and the coefficient of the second term in the q-expansion of ch M^1 represents the dimension of the Lie algebra V_1. So we can use this fact to determine for possible Kac-Moody subVOAs $\tilde{V_1}$ and then we can classify the VOA genera.

5.3.1 The reductive Lie algebras and the affine Kac-Moody Lie algebras and theirs associated VOAs

We need some detail of a reductive Lie algebra and an affine Kac-Moody Lie algebra in order to classify the dimension of the corresponding Kac-Moody subVOAs.

We first give a brief detail of a Lie algebra. A Lie algebra $\mathfrak{g}$ is a vector space equipped with an antisymmetric binary operation $[\cdot,\cdot]$, called a commutator, mapping $\mathfrak{g}\times\mathfrak{g}$ into $\mathfrak{g}$, and further constrained to satisfy the Jacobi identity

$$[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0 \text{ for } X,Y,Z\in\mathfrak{g}.$$

A *simple Lie algebra* over $\mathbb{C}$ is a non-abelian Lie algebra whose only ideals are 0 and itself. A direct sum of simple Lie algebras is called a *semisimple Lie algebra*. There are nine types of simple Lie algebras over $\mathbb{C}$, four infinite series of *classical algebras* and five *exceptional algebras*. The following notation is commonly used:

classical algebras : $A_n(n\geq 1), B_n(n\geq 3), C_n(n\geq 2), D_n(n\geq 4)$

exceptional algebras : G_2, F_4, E_6, E_7, E_8.

The subscript on the designation $A, B, ..., E$ is the rank of the algebra. We give the dimensions of the simple Lie algebras with their corresponding formulas in Tables 5.2 and 5.3. Moreover, the direct sum among these simple Lie algebras has the sum of theirs dimensions as the dimension of the direct sum.

Table 5.2: *The dimensions of the classical simple Lie algebras*

Types	n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A_n	$n(n+2)$	3	8	15	24	35	48	63	80	99	120	143	168	195	224	255	288
B_n	$n(2n+1)$	-	-	21	36	55	78	105	136	171	210	253	300	351	406	465	528
C_n	$n(2n+1)$	-	10	21	36	55	78	105	136	171	210	253	300	351	406	465	528
D_n	$n(2n-1)$	-	-	-	28	45	66	91	120	153	190	231	276	325	378	435	496

Table 5.3: *The dimensions of the exceptional Lie algebras*

Types	G_2	F_4	E_6	E_7	E_8
Dimensions	14	52	78	133	248

The algebras A_n, D_n, and E_n have symmetric Cartan matrices (cf. [KMPS90]). And they have the property that all nonzero roots are of equal length, so the root and coroot lattices are identical (cf. [MS97]), i.e., they correspond to the root lattices described in Section 2.2

A *reductive Lie algebra* is a direct sum of a semisimple Lie algebra and an abelian Lie algebra. The Heisenberg Lie algebra H_1 (see detail in [FBZ04]) is an abelian Lie algebra with dimension 1. The direct sum of any semisimple Lie algebra with dimension m and H_1 is a reductive Lie algebra with dimension $m+1$. Note that the Heisenberg Lie algebra has its associated VOA $\tilde{V_{H_1}}$ with central charge 1 (cf. [FBZ04]).

We define the affine Kac-Moody Lie algebra as a central extension of the formal loop algebra in Section 2.3.4. Let $\mathfrak{g}$ be the simple Lie algebra and $\hat{\mathfrak{g}}$ be the corresponding affine Kac-Moody Lie algebra.

Definition 5.3.1 (cf. [FBZ04]). We say that a vertex operator algebra V has a $\hat{\mathfrak{g}}$-structure of level k denoted by $V_k(\hat{\mathfrak{g}})$ if there is an injection $\alpha : \mathfrak{g} \to V$ such that the Fourier coefficients of the vertex operators $Y(\alpha(A), z)$, $A \in \mathfrak{g}$, generate an action of $\hat{\mathfrak{g}}$ on V of level k.

The VOA $V_k(\mathfrak{g})$ has a natural conformal vector, called the *Segal-Sugawara* vector (cf. [FBZ04]), where $k \neq -h^\vee$, where $h^\vee$ denotes the *dual Coxeter number* of $\mathfrak{g}$. We have an isomorphism of vector spaces

$$V_k(\mathfrak{g}) \simeq U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])v_k.$$

Table 5.4: *The dual Coxeter numbers of the simple Lie algebras*

Types	A_n	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
Dual Coxeter number	$n+1$	$2n-1$	$n+1$	$2n-2$	4	9	12	18	30

Pick a basis $\{J^a\}_{a=1,\ldots,d}$ of $\mathfrak{g}$ (where $d = \dim \mathfrak{g}$), and let $\{J_a\}_{a=1,\ldots,d}$ be its dual basis with respect to the invariant bilinear form $(\cdot,\cdot)$ (cf. [FBZ04]).

We write

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1},\ J_a(z) = \sum_{n\in\mathbb{Z}} J_{a,n} z^{-1-1}.$$

Set

$$S = \frac{1}{2}\sum_{a=1}^{d} J_{a,-1}J^a_{-1}v_k.$$

Then for $k \neq h^\vee$, $S/(k+h^\vee)$ is a conformal vector in $V_k(\mathfrak{g})$, of central charge

$$c(k) = \frac{k\cdot\dim(\mathfrak{g})}{k+h^\vee}. \tag{5.3.1}$$

From (5.3.1), we have the formula in term of the dimension of $\mathfrak{g}$

$$\dim\,\mathfrak{g} = \frac{c(k)(k+h^\vee)}{k}.$$

Recall that the component V_1 of the VOA V has a structure of a reductive Lie algebra. By this fact and since the VOAs here are unitary, we can classify the possible Kac-Moody subVOAs $\tilde{V_1}$ of the VOAs arising from some small MTCs.

Example 5.3.2. The VOA genus $\mathcal{G}(\mathbf{qs_2}, \mathbf{1})$.

We have from Example 5.2.1 that the dimension of the corresponding reductive Lie algebra V_1 of qs_2 is 3. By computation, $A_{1,1}$ is the only possible affine Kac-Moody Lie algebra corresponding to dimension 3 and central charge 1.

Theorem 5.3.3. *The possible Kac-Moody subVOA $\tilde{V_1}$ of the VOA V in $\mathcal{G}(qs_2, 1)$ is $A_{1,1}$.*

Example 5.3.4. The VOA genus $\mathcal{G}(\mathbf{qs_2} \otimes \mathbf{qs_2}, \mathbf{10})$.

We have from Example 5.2.3 that the dimension of the corresponding reductive Lie algebra V_1 of $qs_2 \otimes qs_2$ is $190 + 32d$ where d is a nonnegative integer. In this case, we need to find the value of the parameter d and hence we will have the dimension.

Recall that the central charge of the Kac-Moody VOA of a simple Lie algebra $\mathfrak{g}$ at level k is $c(k) = \dfrac{k \cdot \dim(\mathfrak{g})}{k + h^\vee}$, where $h^\vee$ is the dual Coxeter nubmer of $\mathfrak{g}$. We can estimate the highest possible dimension of the Lie algebras using the central charge c as the rank of the (semi)simple Lie algebras of level 1.

In this case, the highest possible dimension arises as the dimension of the simple Lie algebra E_8 and another simple Lie algebra of rank 2. C_2 has the highest dimension among the rank 2 Lie algebras. So the highest dimension could be 258. Hence d could be 0, 1, 2, or 3 and the possible dimensions are 190, 222, 254, and 286.

By computation, we get the possible affine Kac-Moody subVOAs as in the following table:

Dimension	Affine Kac-Moody subVOAs
190	$D_{10,1}$
222	None
254	$A_{1,1} \otimes A_{1,1} \otimes E_{8,1}$
286	None

Theorem 5.3.5. *The possible Kac-Moody subVOA $\tilde{V_1}$ of the VOAs V in $\mathcal{G}(qs_2 \otimes qs_2, 10)$ are $D_{10,1}$ and $A_{1,1} \otimes A_{1,1} \otimes E_{8,1}$.*

Example 5.3.6. The VOA genus $\mathcal{G}(\mathbf{Ising1}, \mathbf{17/2})$.

The MTC $Ising1$ is of rank 3 with conformal weights $h_1 = 0, h_2 = 1/2$, and $h_3 = 1/16$. From Table B.3, the dimension of corresponding reductive Lie algebra V_1 is $136+112d$ where d is a nonnegative integer.

By computation, we get the possible Kac-Moody subVOAs as in the following table:

Dimension	Kac-Moody subVOAs
136	$B_{8,1},\ A_{1,1} \otimes E_{7,1}(1/2),\ A_{1,2} \otimes E_{7,1}$
248	$E_{8,1}(1/2)$
360	None

Note that sometime the central charges of a Kac-Moody subVOA in the resulting tensor product do not add up to c but the sum of the dimensions already added up. We use the notation (1/2) as the remainder component with the remainder central charge.

Theorem 5.3.7. *The possible Kac-Moody subVOA $\tilde{V}_1$ of the VOA V in $\mathcal{G}(Ising1, 17/2)$ are $B_{8,1}$, $A_{1,1} \otimes E_{7,1} \otimes L_{1/2}(0)$, $A_{1,2} \otimes E_{7,1}$, and $E_{8,1} \otimes L_{1/2}(0)$.*

Note that there is no Kac-Moody subVOA in the genus $\mathcal{G}(3fieldsx, 8/7)$, since the character ch M^1 of the MTC $3fieldsx$ has a pole order larger than 1. So ch M^1 cannot be a character of a VOA.

By similar computation for each small MTC, we have the following theorem.

Theorem 5.3.8. *Table B.4 consists of the possible Kac-Moody subVOAs $\tilde{V}_1$ of the VOA V in each genus $\mathcal{G}(\mathcal{C}, c)$.*

Note that in Table B.4, the number in the parenthesis represents the remainder of the central charge in the resulting tensor product.

Remark: Our method does not work with the cases which the remainder of the central charge larger than 1, since the information of these cases are unknown. So we only give a couple examples of the possible subVOAs $\tilde{V}_1$ and the number of the rest of the subVOAs.

5.3.2 The genera $\mathcal{G}(\mathcal{C}, c)$

Table B.4 contains all of the possible Kac-Moody subVOAs $\tilde{V_1}$ of the VOA V in each genus $\mathcal{G}(\mathcal{C}, c)$. We can classify the VOAs V in the genus by determining the corresponding rank and the corresponding conformal weights of the possible Kac-Moody subVOAs $\tilde{V_1}$ in Table B.4.

Example 5.3.9. The VOA genus $\mathcal{G}(qs_2, 1)$.

From Theorem 5.3.3, the only possible Kac-Moody subVOAs $\tilde{V_1}$ in the genus $\mathcal{G}(qs_2, 1)$ is $A_{1,1}$. Hence we have the following theorem.

Theorem 5.3.10. *The VOA $A_{1,1}$ is up to isomorphism the only VOA in $\mathcal{G}(qs_2, 1)$.*

Proof : The MTC of the VOA $\tilde{V_1} \cong A_{1,1}$ is already qs_2 and $A_{1,1}$ has central charge 1. **q.e.d.**

Example 5.3.11. The VOA genus $\mathcal{G}(qs_2 \otimes qs_2, 10)$.

From Theorem 5.3.5, the possible Kac-Moody subVOAs $\tilde{V_1}$ in the genus $\mathcal{G}(qs_2 \otimes qs_2, 10)$ are $D_{10,1}$ and $A_{1,1} \otimes A_{1,1} \otimes E_{8,1}$ and hence we have the following theorem.

Theorem 5.3.12. *The two VOAs $D_{10,1}$ and $A_{1,1} \otimes A_{1,1} \otimes E_{8,1}$ are up to isomorphism all the VOAs in $\mathcal{G}(qs_2 \otimes qs_2, 10)$.*

Proof : The MTCs of the VOAs $\tilde{V_1} \cong D_{10,1}$, and $\tilde{V_1} \cong A_{1,1} \otimes A_{1,1} \otimes E_{8,1}$ are already $qs_2 \otimes qs_2$ and in each case $\tilde{V_1}$ has central charge 10. **q.e.d.**

Remarks:

- We denote the method we use in the above examples "method 1". This method is used in the cases that the MTC of the possible Kac-Moody subVOAs $\tilde{V_1}$ in a genus $\mathcal{G}(\mathcal{C}, c)$ are already $\mathcal{C}$.

- We use the information such as conformal weights, central charge, the corresponding MTC etc. of each Kac-Moody subVOA $\tilde{V_1}$ from the database [Dat] to determine our results.

The following genera are classified by applying method 1: $\mathcal{G}(t_m, 8)$, $\mathcal{G}(qs_2, 9)$, $\mathcal{G}(\overline{qs_2}, 7)$, $\mathcal{G}(LY, 14/5)$, $\mathcal{G}(LY, 54/5)$, $\mathcal{G}(\overline{LY}, 26/5)$, $\mathcal{G}(qs_3, 2)$, $\mathcal{G}(qs_3, 10)$, $\mathcal{G}(\overline{qs_3}, 6)$, $\mathcal{G}(\overline{qs_3}, 14)$, $\mathcal{G}(\overline{Ising1}, 15/2)$, $\mathcal{G}(Ising3, 5/2)$, $\mathcal{G}(Ising3, 21/2)$, $\mathcal{G}(\overline{Ising3}, 11/2)$, $\mathcal{G}(Ising4, 7/2)$, $\mathcal{G}(Ising4, 23/2)$, $\mathcal{G}(\overline{Ising4}, 9/2)$, $\mathcal{G}(\overline{qs_4}, 7)$, $\mathcal{G}(qn_4, 5)$, $\mathcal{G}(\overline{qn_4}, 3)$, $\mathcal{G}(\overline{qn_4}, 11)$, $\mathcal{G}(qu_2, 8)$, $\mathcal{G}(qv_2, 4)$, $\mathcal{G}(qv_2, 12)$, $\mathcal{G}(qs_2 \otimes qs_2, 2)$, $\mathcal{G}(\overline{qs_2} \otimes \overline{qs_2}, 6)$, $\mathcal{G}(qs_2 \otimes \overline{qs_2}, 8)$, $\mathcal{G}(qs_2 \otimes LY, 19/5)$, and $\mathcal{G}(LY \otimes LY, 28/5)$.

Before we can give the next example, we need the following notions. For a VOA V with an *Ising vector* e of V, we define the commutant subalgebra $\mathrm{Com}_V(e) := \{A \in V \mid e_{(0)}A = 0\}$. Let W be a unitary commutant subVOA $\tilde{V_1}$ of V, i.e., $W = \mathrm{Com}_V(\tilde{V_1})$. We have $c(W) = c(V) - c(\tilde{V_1})$, where $c(W), c(V)$, and $c(\tilde{V_1})$ are the central charges of W, V, and $\tilde{V_1}$ respectively and $\tilde{V_1}$ is a subVOA of V.

Theorem 5.3.13. *Let W be a unitary VOA of central charge $c < 1$. Then W is isomorphic to an extension of the Virasoro minimal model VOA $L_c(0)$, where $c = 1 - \dfrac{6}{p(p+1)}$ for $p = 2, 3, 4, ...$*

The central charge c in the above theorem belongs to the *minimal series* which the first few elements are 0, 1/2, 7/10, 4/5, 6/7, ...

Remark: The genus $\mathcal{G}(Ising1, 1/2)$ contains only the minimal model (the Ising model) $L_{1/2}(0)$. So it is the only VOA in this genus.

Theorem 5.3.14. *The only VOA up to isomorphism in $\mathcal{G}(Ising1, 1/2)$ is $L_{1/2}(0)$.*

Example 5.3.15. The VOA genus $\mathcal{G}(Ising1, 17/2)$.

From Theorem 5.3.7, the possible subVOAs $\tilde{V}_1 \otimes W$ in the genus $\mathcal{G}(Ising1, 17/2)$ are $B_{8,1}$, $A_{1,1} \otimes E_{7,1} \otimes L_{1/2}(0)$, $A_{1,2} \otimes E_{7,1}$, and $E_{8,1} \otimes L_{1/2}(0)$.

The MTCs of $B_{8,1}$ and $E_{8,1} \otimes L_{1/2}(0)$ have the ranks and conformal weights of the MTC of this genus. The MTCs of $A_{1,1} \otimes E_{7,1} \otimes L_{1/2}(0)$ and $A_{1,2} \otimes E_{7,1}$ have ranks larger than 4. But these can be the VOA-extensions.

We apply the idea of the simple current extension (cf. [Höh03]) to determine our result. Let V be a rational VOAs. We call a VOA (W, Y_W) an *extension* of V if it contains a subVOA isomorphic to V and has the same vacuum and Virasoro element as V. The VOA-extensions W of a rational VOA V satisfying some certain conditions such as the conditions in Theorem 3.3.1 can be determined completely in terms of the associated MTC (cf. Theorem 4.2 in [Höh03]).

A simple module M_i is called a *simple current* if for each simple module M_j there is another simple module $M_{j'}$ such that $M_i \times M_j = M_{j'}$ holds in the fusion algebra.

Theorem 5.3.16. *[cf. [Höh03]] Let V be a rational VOA which has an abelian intertwining operator algebra structure on the direct sum of the simple current. Let C be a subgroup of the abelian group $A \subset I$ of labels of the simple currents for which the modules M_c, $c \in C$, have integral conformal weight. Then there exists a unique simple VOA-extension (W, Y_W) of the form $W \cong \bigoplus_{c \in C} n_c M_c$, $n_c \geq 0$, and one has $n_c = 1$.*

By the notion of the VOA extension and simple current, if a module has integral conformal weight, then it can be an extension of the VOA. So we can use this fact to determine our results by looking at the sum of the appropriate corresponding conformal weights. So we have to consider the rank and conformal weights of the extended VOA whether it is in the genus or not.

Consider $A_{1,1} \otimes E_{7,1} \otimes L_{1/2}(0)$, the corresponding conformal weights of $A_{1,1}, E_{7,1}$, and $L_{1/2}(0)$ are $\{0,\ 1/4\}$, $\{0,\ 3/4\}$, and $\{0, 1/2, 1/16\}$ respectively. (By combining 1/4 and 3/4 together, the result is equal to 1.) $(1/4, 3/4, 0)$ is the only conformal weight of the MTC of $A_{1,1} \otimes E_{7,1} \otimes L_{1/2}(0)$ which is integral. So $(1/4, 3/4, 0)$ represents a module with integral conformal weights and it is a simple current. The extension by $((1/4, 3/4), 0)$ is isomorphic to $E_{8,1} \otimes L_{1/2}(0)$ which we already have in this genus.

Consider $A_{1,2} \otimes E_{7,1}$, $A_{1,2}$ has conformal weights $\{0,\ 1/2,\ 3/16\}$. There is no integral value for the sum between the elements in $\{0,\ 1/2,\ 3/16\}$ and the elements of $\{0,\ 3/4\}$ which are the conformal weights of $E_{7,1}$. So there is no VOA-extension and hence no extension of the VOA belong to this genus.

Theorem 5.3.17. *The VOAs up to isomorphism in $\mathcal{G}(Ising1, 17/2)$ are $B_{8,1}$ and $E_{8,1} \otimes L_{1/2}(0)$.*

Remarks:

- We call the method we use in the above example "method 2". This method is to determine whether a VOA $\tilde{V_1} \otimes W$ has a VOA extension or not by applying the notion of simple current.

- Since W is a minimal model, we need to determine only the case that the central charge c of $L_c(0)$ is in the minimal series.

Method 2 is applied to all cases which the only simple objects of the MTC of $\tilde{V_1} \otimes W$ of conformal weight 0 (mod 1) are simple currents.

This method can be applied to the following genera $\mathcal{G}(t_m, 16)$, $\mathcal{G}(Ising1, 17/2)$, $\mathcal{G}(Ising2, 3/2)$, $\mathcal{G}(Ising2, 19/2)$, $\mathcal{G}(\overline{Ising2}, 13/2)$, $\mathcal{G}(\overline{Ising4}, 25/2)$, $\mathcal{G}(3fieldsx, 64/7)$, $\mathcal{G}(\overline{qs_2} \otimes LY, 9/5)$, $\mathcal{G}(\overline{qs_2} \otimes LY, 49/5)$, $\mathcal{G}(qs_2 \otimes \overline{LY}, 31/5)$, $\mathcal{G}(\overline{qs_2} \otimes \overline{LY}, 21/5)$, $\mathcal{G}(LY \otimes \overline{LY}, 8)$, $\mathcal{G}(\overline{LY} \otimes \overline{LY}, 12/5)$, $\mathcal{G}(\overline{LY} \otimes \overline{LY}, 52/5)$, $\mathcal{G}(4fieldsx, 10/3)$, and $\mathcal{G}(\overline{4fieldsx}, 14/3)$.

The following table shows the results from the computation regarding of a simple current and a VOA extension.

Note that we also give the information regarding of a module with integral conformal weight and a simple current corresponding to the following genera in the table; $\mathcal{G}(\overline{LY}, 66/5)$, $\mathcal{G}(qs_2 \otimes LY, 59/5)$, $\mathcal{G}(\overline{3fieldsx}, 48/7)$, $\mathcal{G}(qs_2 \otimes \overline{LY}, 71/5)$, and $\mathcal{G}(\overline{qs_2} \otimes \overline{LY}, 61/5)$. Method 2 is not applied to these cases directly.

Table 5.5: Simple current testing results

$\mathcal{G}(\mathcal{C}, c)$	$\tilde{V_1} \otimes W$	conformal weights	integral	s.c.	≥ 2	$(\tilde{V_1} \otimes W)^+$
$\mathcal{G}(t_m, 16)$	$D_{16,1}$	$D_{16,1}$: $0, 2_a, 1/2, 2_b$	(2_a)	yes	yes	yes
			(2_b)	yes	yes	yes
$\mathcal{G}(\overline{LY}, 66/5)^*$	$B_{12,1} \otimes L_{7/10}(0)$	$B_{12,1}$: 0, 1/2, 25/16	(1/2, 3/2)	yes	yes	no
		$L_{7/10}(0)$: 0, 7/16, 3/2, 3/80, 3/5, 1/10	(25/16, 7/16)	no	yes	-
$\mathcal{G}(Ising1, 17/2)$	$A_{1,1} \otimes E_{7,1} \otimes L_{1/2}$	$A_{1,1}$: 0, 1/4	(1/4, 3/4, 0)	yes	no	-
		$E_{7,1}$: 0, 3/4				
		$L_{1/2}(0)$: 0, 1/2, 1/16				
	$A_{1,2} \otimes E_{7,1}$	$A_{1,2}$: 0, 1/2, 3/16	-	-	-	-
		$E_{7,1}$: 0, 3/4				
$\mathcal{G}(Ising2, 3/2)$	$A_{1,1} \otimes L_{1/2}(0)$	$A_{1,1}$: 0, 1/4	-	-		-
		$L_{1/2}(0)$: 0, 1/2, 1/16				
$\mathcal{G}(Ising2, 19/2)$	$A_{1,1} \otimes E_{8,1} \otimes L_{1/2}(0)$	same as in $\mathcal{G}(Ising2, 3/2)$	-	-	-	-
$\mathcal{G}(\overline{Ising2}, 13/2)$	$E_{6,1} \otimes L_{1/2}(0)$	$E_{6,1}$: 0, 2/3, 2/3	-	-	-	-
		$L_{1/2}(0)$: 0, 1/2, 1/16				
$\mathcal{G}(\overline{Ising4}, 25/2)$	$D_{12,1} \otimes L_{1/2}(0)$	$D_{12,1}$: $0, 1/2_a, 3/2_b, 3/2_c$	$(1/2_a, 1/2)$	yes	no	-
		$L_{1/2}(0)$: 0, 1/2, 1/16	$(3/2_b, 1/2)$	yes	yes	yes
			$(3/2_c, 1/2)$	yes	yes	yes
$\mathcal{G}(3fieldsx, 64/7)$	$A_{1,5} \otimes E_{7,1}$	$A_{1,5}$: 0, 5/4, 6/7, 3/28, 2/7, 15/28	(5/4, 3/4)	yes	yes	yes
		$E_{7,1}$: 0, 3/4				
$\mathcal{G}(\overline{3fieldsx}, 48/7)^*$	$E_{6,1} \otimes L_{6/7}(0)$	$E_{6,1}$: $0, 2/3_a, 2/3_b$	$(2/3_a, 4/3)$	no	yes	
		$L_{6/7}(0)$: 0, 3/8, 4/3, 23/8, 5,	$(2/3_b, 4/3)$	no	yes	
		1/56, 10/21,85/56, 22/7, 1/21,	(0,5)	yes	yes	
		33/56,12/7, 5/56, 5/7, 1/7				
$\mathcal{G}(qs_2 \otimes LY, 59/5)^*$	$A_{1,1} \otimes D_{10,1} \otimes L_{4/5}(0)$	$A_{1,1}$: 0, 1/4	(0,0,3)	yes	yes	no
		$D_{10,1}$: 0, 5/4, 1/2, 5/4				
		$L_{4/5}(0)$: 0, 2/5, 7/5, 3, 1/40, 21/40, 13/8, 1/15, 2/3, 1/8				
	$A_{1,3} \otimes D_{10,1}$	$A_{1,3}$: 0, 3/4, 2/5, 3/20	$(3/4, 5/4_a)$	yes	yes	yes

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Table 5.5 – *Continued from previous page*

$\mathcal{G}(\mathcal{C}, c)$	$\tilde{V_1} \otimes W$	conformal weights	integral	s.c.	≥ 2	$(\tilde{V_1} \otimes W)^+$
		$D_{10,1}$: $0, 5/4_a, 1/2, 5/4_b$	$(3/4, 5/4_b)$	yes	yes	yes
$\mathcal{G}(\overline{qs_2} \otimes LY, 9/5)$	$A_{1,1} \otimes L_{4/5}(0)$	$A_{1,1}$: 0, 1/4	(0, 3)	yes	yes	no
		$L_{4/5}(0)$: 0, 2/5, 7/5, 3, 1/40,				but no
		21/40, 13/8, 1/15, 2/3, 1/8				extension
$\mathcal{G}(\overline{qs_2} \otimes LY, 49/5)$	$A_{1,1} \otimes E_{8,1} \otimes L_{4/5}(0)$	same as in $\mathcal{G}(\overline{qs_2} \otimes LY, 9/5)$	(0, 0, 3)	yes	yes	no
						but no
						extension
$\mathcal{G}(qs_2 \otimes \overline{LY}, 31/5)$	$B_{5,1} \otimes L_{7/10}(0)$	$B_{5,1}$: 0, 1/2, 11/16	(1/2, 3/2)	yes	yes	no
		$L_{7/10}(0)$: 0, 7/16, 3/2, 3/80,				but no
		3/5, 1/10				extension
$\mathcal{G}(qs_2 \otimes \overline{LY}, 71/5)^*$	$B_{3,1} \otimes D_{10,1} \otimes L_{7/10}(0)$	$B_{3,1}$: $0, 1/2_a, 7/16$	$(1/2_a, 0, 3/2)$	yes	yes	
		$D_{10,1}$: $0, 5/4, 1/2_b, 5/4$	$(0, 1/2_b, 3/2)$	yes	yes	
		$L_{7/10}(0)$: 0, 7/16, 3/2, 3/80,	$(1/2_a, 1/2_b, 0)$	yes	no	
		3/5, 1/10				
	$B_{6,1} \otimes E_{7,1} \otimes L_{7/10}(0)$	$B_{6,1}$: 0, 1/2, 13/16	(1/2, 0, 3/2)	yes	yes	
		$E_{7,1}$: 0, 3/4	(13/16, 3/4, 7/16)	no	yes	
		$L_{7/10}(0)$: 0, 7/16, 3/2, 3/80,				
		3/5, 1/10				
	$C_{3,1} \otimes D_{10,1}$	$C_{3,1}$: 0, 3/4, 3/5, 7/20	$(3/4, 5/4_a)$	yes	yes	
		$D_{10,1}$: $0, 5/4_a, 1/2, 5/4_b$	$(3/4, 5/4_b)$	yes	yes	
	$A_{1,1} \otimes B_{11,1}$	$A_{1,1}$: 0, 1/4	not			
	$\otimes \tilde{V}_{H_1} \otimes L_{7/10}(0)$	$B_{11,1}$: $0, 1/2_a, 7/16$	applicable			
		H_1: -				
		$L_{7/10}(0)$: 0, 7/16, 3/2, 3/80,				
		3/5, 1/10				
	$A_{1,1} \otimes B_{12,1} \otimes L_{7/10}(0)$	$A_{1,1}$: 0, 1/4	(0, 1/2, 3/2)	yes	yes	
		$B_{12,1}$: 0, 1/2, 25/16	(0, 25/16, 7/16)	no	yes	
		$L_{7/10}(0)$: 0, 7/16, 3/2, 3/80,				
		3/5, 1/10				
	$B_{5,1} \otimes E_{8,1} \otimes L_{7/10}(0)$	same as in $\mathcal{G}(qs_2 \otimes \overline{LY}, 31/5)$	(1/2, 3/2, 0)	yes	yes	
$\mathcal{G}(\overline{qs_2} \otimes \overline{LY}, 21/5)$	$B_{3,1} \otimes L_{7/10}(0)$	$B_{3,1}$: 0, 1/2, 7/16	(1/2, 3/2)	yes	yes	no
		$L_{7/10}(0)$: 0, 7/16, 3/2, 3/80,				but no
		3/5, 1/10				extension
$\mathcal{G}(\overline{qs_2} \otimes \overline{LY}, 61/5)^*$	$A_{1,1} \otimes B_{10,1} \otimes L_{7/10}(0)$	$A_{1,1}$: 0, 1/4	(0, 1/2, 3/2)	yes	yes	
		$B_{10,1}$: 0, 1/2, 21/16	(1/4, 21/16, 7/16)	no	yes	
		$L_{7/10}(0)$: 0, 7/16, 3/2, 3/80,				

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Table 5.5 – *Continued from previous page*

$\mathcal{G}(\mathcal{C}, c)$	$\tilde{V_1} \otimes W$	conformal weights	integral	s.c.	≥ 2	$(\tilde{V_1} \otimes W)^+$
		3/5, 1/10				
	$B_{3,1} \otimes E_{8,1} \otimes L_{7/10}(0)$	same as in $\mathcal{G}(\overline{qs_2} \otimes \overline{LY}, 21/5)$	(1/2, 3/2, 0)	yes	yes	
$\mathcal{G}(LY \otimes \overline{LY}, 8)$	$A_{1,1} \otimes A_{7,1}$	$A_{1,1}$: 0, 1/4	$(1/4, 3/4_a)$	yes	no	-
		$A_{7,1}$: $0, 7/16, 3/4_a, 15/16,$	$(1/4, 3/4_b)$	yes	no	-
		$1, 15/16, 3/4_b, 7/16$				
$\mathcal{G}(\overline{LY} \otimes \overline{LY}, 12/5)$	$A_{1,8}$	$A_{1,8}$: 0, 2, 3/40, 63/40, 1/5,	(2)	yes	yes	yes
		6/5, 3/8, 7/8, 3/5				
$\mathcal{G}(\overline{LY} \otimes \overline{LY}, 52/5)$	$A_{1,8} \otimes E_{8,1}$	same as in $\mathcal{G}(\overline{LY} \otimes \overline{LY}, 12/5)$	(2, 0)	yes	yes	yes
$\mathcal{G}(4fieldsx, 10/3)$	$A_{1,1} \otimes A_{1,7}$	$A_{1,1}$: 0, 1/4	(1/4, 7/4)	yes	yes	yes
		$A_{1,7}$: 0, 7/4, 4/3, 1/12, 2/9,				
		35/36, 2/3, 5/12				

Note that the fifth column in the table above contains the simple objects of the MTC of $\tilde{V_1} \otimes W$ which have integral conformal weights. The sixth column shows whether the simple objects in column fifth are simple currents or not. The seventh column shows whether the conformal weights of the simple objects in column fifth are larger than 1 or not. And the last column shows whether the MTC of the extension $(\tilde{V_1} \otimes W)^+$ is $\mathcal{C}$ or not. We use the computer algebra software **Kac** to compute the simple current extensions. The source codes are in appendix C.

Remarks:

1. Each of the possible subVOA in the genus $\mathcal{G}(\overline{4fieldsx}, 14/3)$ contains no minimal model $L_c(0)$, i.e., the central charge c of $L_c(0)$ is not in the minimal series. So there is no subVOA $\tilde{V_1}$ of the VOA in this genus to be considered.

2. In $\mathcal{G}(t_m, 16)$, there are two simple currents but they are isomorphic to the lattice VOA D_{16}^+. Since $\tilde{V_1}$ is isomorphic to a lattice VOA, any extension of $\tilde{V_1}$ is again a lattice VOA. Hence the VOA is $D_{16,1}^+$.

3. If the conformal weight of a module of $\tilde{V_1} \otimes W$ is 1, then the Lie algebra of an extension

would be larger than V_1. So this can be ignored. This applies to the cases with "no" in the sixth column.

4. For the cases with "no" in the fifth column and "yes" in the sixth column, we cannot determine these cases. We mark these cases with $\mathcal{G}(\mathcal{C}, c)^*$.

5. In the cases with "no" in the last column, there are two possibilities. First, the MTC of $(\tilde{V_1} \otimes W)^+$ is not $\mathcal{C}$ and the family of the conformal weights of the MTC does not contain all of the conformal weights of $\mathcal{C}$. So there is no other extension which corresponds to the MTC $\mathcal{C}$. Second, the MTC of $(\tilde{V_1} \otimes W)^+$ is not $\mathcal{C}$ but its family of the conformal weights contains all of the conformal weights of $\mathcal{C}$. So there might be other extensions in this case and we cannot determine this case. We again mark this case with $\mathcal{G}(\mathcal{C}, c)^*$.

The rest of the genera can be determined by the methods above and some further information which we will describe them case by case. We apply the idea of the lattice VOA (cf. 2.3.5) to determine the case below. We call this method regarding the lattice VOA "method 3".

Method 3 is applied to the genus $\mathcal{G}(\overline{qs_2}, 15)$. The possible subVOAs $\tilde{V_1}$ in this genus are $E_{7,1} \otimes E_{8,1}$ and $A_{1,1} \otimes D_{14,1}$. The lattices $E_7 \oplus E_8$ and $A_1 \oplus D_{14}^+$ belong to the correct lattice genus. Hence their associated VOAs belong to this genus.

Theorem 5.3.18. *The VOAs up to isomorphism in $\mathcal{G}(\overline{qs_2}, 15)$ are $E_{7,1} \otimes E_{8,1}$, $A_{1,1} \otimes D_{14,1}^+$.*

Example 5.3.19. The VOA genus $\mathcal{G}(qs_4, 1)$.

From Table B.4, the possible subVOAs $\tilde{V_1}$ in $\mathcal{G}(qs_4, 1)$ is $\tilde{V}_{H_1}$. This implies that an extension of $\tilde{V_1}$ is a lattice VOA. The only lattice in the corresponding genus is D_1.

Theorem 5.3.20. *The VOA up to isomorphism in $\mathcal{G}(qs_4, 1)$ is $D_{1,1}$.*

We will call this method "method 3*", since it also apply the idea of a lattice VOA. The method is applied to the genera $\mathcal{G}(qs_4, 9)$, $\mathcal{G}(qn_4, 13)$, and $\mathcal{G}(\overline{qs_2} \otimes \overline{qs_2}, 14)$.

The possible subVOAs $\tilde{V}_1$ of a VOA in $\mathcal{G}(qs_4, 9)$ are $\tilde{V}_{H_1} \otimes E_{8,1}$ and $D_{9,1}$. The MTC of $D_{9,1}$ is already qs_4. Moreover, the subVOA $\tilde{V}_{H_1} \otimes E_{8,1}$ gives the VOA $D_{1,1} \otimes E_{8,1}$.

Theorem 5.3.21. *The VOAs up to isomorphism in $\mathcal{G}(qs_4, 9)$ are $D_{1,1} \otimes E_{8,1}$ and $D_{9,1}$.*

The possible subVOAs $\tilde{V}_1$ of a VOA in $\mathcal{G}(qn_4, 13)$ are $\tilde{V}_{H_1} \otimes D_{12,1}$, $A_{1,1} \otimes D_{12,1}$, $D_{5,1} \otimes E_{8,1}$, and $D_{13,1}$. The MTCs of $D_{5,1} \otimes E_{8,1}$ and $D_{13,1}$ are already qn_4, so the VOAs $D_{5,1} \otimes E_{8,1}$ and $D_{13,1}$ are in this genus. Hence a VOA V has again to be a lattice VOA. The lattices for the genus $\mathcal{G}(qn_4, 13)$ are $D_5 \oplus E_8$, and D_{13}.

Theorem 5.3.22. *The VOAs up to isomorphism in $\mathcal{G}(qn_4, 13)$ are $D_{5,1} \otimes E_{8,1}$ and $D_{13,1}$.*

Now consider the genus $\mathcal{G}(\overline{qs_2} \otimes \overline{qs_2}, 14)$. The possible subVOAs $\tilde{V}_1$ of a VOA in $\mathcal{G}(\overline{qs_2} \otimes \overline{qs_2}, 14)$ are $E_{7,1}^{\otimes 2}$, $H_1^{\otimes 2} \otimes D_{12,1}$, $A_{1,1}^{\otimes 2} \otimes D_{12,1}$, $D_{6,1} \otimes E_{8,1}$, $D_{14,1}$, and $H_1 \otimes D_{13,1}$. The MTC of $E_{7,1}^{\otimes 2}$, $D_{6,1} \otimes E_{8,1}$, and $D_{14,1}$ are already $\overline{qs_2} \otimes \overline{qs_2}$. So the VOAs $E_{7,1}^{\otimes 2}$, $D_{6,1} \otimes E_{8,1}$, and $D_{14,1}$ are in this genus. Again, a VOA V has to be a lattice VOA. The lattices for the genus $\mathcal{G}(\overline{qs_2} \otimes \overline{qs_2}, 14)$ are $E_7^{\oplus 2}$, $(A_1^{\oplus 2} \oplus D_{12})^+$, $D_6 \oplus E_8$, $D_1^{\oplus 2} \oplus D_{12}$, and D_{14}.

Theorem 5.3.23. *The VOAs up to isomorphism in $\mathcal{G}(\overline{qs_2} \otimes \overline{qs_2}, 14)$ are $E_{7,1}^{\otimes 2}$, $(A_{1,1}^{\otimes 2} \otimes D_{12,1})^+$, $D_{6,1} \otimes E_{8,1}$, $D_{14,1}$, and $D_{1,1}^{\otimes 2} \otimes D_{12,1}$.*

Finally, the genus $\mathcal{G}(3fieldsx, 8/7)$ contains no VOA, since there is no Kac-Moody subVOA $\tilde{V}_1$ of the VOA in this genus.

So by applying the methods explained above to each small MTC, we have the following theorem.

Theorem 5.3.24. *The genera of the VOAs arising from the small MTCs with central charge at most 16 are classified in Table B.5.*

Note that in Table B.5, $(VOA)^+$ represents a VOA extension. The number in the last column represents the method for classifying the VOAs in each genus:

1 - method 1.

2 - method 2.

3 - method 3.

3^* - method 3 with the extension of $\tilde{V}_{H_1}$.

4 - special case for $\mathcal{G}(3feildsx, 8/7)$.

Finally, we will classify which genera $\mathcal{G}(\mathcal{C}, c)$ from Table B.5 are code type. We need the following definition.

Definition 5.3.25. A VOA genus is said to be *code type* if each VOA in the genus is a lattice VOA and each associated lattice belongs to the same code type lattice genus.

So the following VOA genera from Table B.5 are code type: $\mathcal{G}(t_m, 8)$, $\mathcal{G}(t_m, 16)$, $\mathcal{G}(qs_2, 1)$, $\mathcal{G}(qs_2, 9)$, $\mathcal{G}(\overline{qs_2}, 7)$, $\mathcal{G}(\overline{qs_2}, 15)$, $\mathcal{G}(qu_2, 8)$, $\mathcal{G}(qv_2, 4)$, $\mathcal{G}(qv_2, 12)$, $\mathcal{G}(qs_2 \otimes qs_2, 2)$, $\mathcal{G}(qs_2 \otimes qs_2, 10)$, $\mathcal{G}(\overline{qs_2} \otimes \overline{qs_2}, 6)$, and $\mathcal{G}(qs_2 \otimes \overline{qs_2}, 8)$.

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Appendix A

Code and Lattice Data

In order to prove Proposition 4.2.7 and Theorem 4.2.8, we have to compare the number of the lattices in each genus with the number of the doubly even binary codes in each corresponding genus. We can do it in two ways: direct and indirect computations.

A.1 Direct computation

We can compute the number of lattices in one particular genus directly by the command ' ♯ ' followed by lattice genus name from Magma.

First, we begin by constructing a list of the basic doubly even binary codes starting from length 1. And we get the basic codes as the following table:

Table A.1: The basic doubly even codes

Notation	**Generating vectors**	**Description**
t_n	[0,0,...,0]	a zero dimensional code of length n
d_4	[1,1,1,1]	a one dimensional code of length 4
d_6	[0,0,1,1,1,1],[1,1,1,1,0,0]	a two dimensional code of length 6

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Table A.1 – *Continued from previous page*

Notation	**Generating vectors**	**Description**
e_7	[0,0,0,1,1,1,1],[0,1,1,1,1,0,0], [1,0,1,0,1,0,1]	a three dimensional code of length 7
h_8	[1,1,1,1,1,1,1,1]	a one dimensional code of length 8
d_8	[0,0,0,0,1,1,1,1],[0,0,1,1,1,1,0,0], [1,1,1,1,0,0,0,0]	a three dimensional code of length 8
e_8	[0,0,0,0,1,1,1,1],[0,0,1,1,1,1,0,0], [1,1,1,1,0,0,0,0],[0,1,0,1,0,1,0,1]	a four dimensional code of length 8
d_{10}	[0,0,0,0,0,0,1,1,1,1],[0,0,0,0,1,1,1,1,0,0], [0,0,1,1,1,1,0,0,0,0],[1,1,1,1,0,0,0,0,0,0]	a four dimensional code of length 10
h_{12}	[1,1,1,1,1,1,1,1,1,1,1,1]	a one dimensional code of length 12
d_{12}	[0,0,0,0,0,0,0,0,1,1,1,1], [0,0,0,0,0,0,1,1,1,1,0,0], [0,0,0,0,1,1,1,1,0,0,0,0], [0,0,1,1,1,1,0,0,0,0,0,0], [1,1,1,1,0,0,0,0,0,0,0,0]	a five dimensional code of length 12
d_{14}	[0,0,0,0,0,0,0,0,0,0,1,1,1,1], [0,0,0,0,0,0,0,0,1,1,1,1,0,0], [0,0,0,0,0,0,1,1,1,1,0,0,0,0], [0,0,0,0,1,1,1,1,0,0,0,0,0,0], [0,0,1,1,1,1,0,0,0,0,0,0,0,0], [1,1,1,1,0,0,0,0,0,0,0,0,0,0]	a six dimensional code of length 14
e_{15}	[0,0,0,0,0,0,0,0,0,0,0,1,1,1,1], [0,0,0,0,0,0,0,0,0,1,1,1,1,0,0], [0,0,0,0,0,0,0,1,1,1,1,0,0,0,0], [0,0,0,0,0,1,1,1,1,0,0,0,0,0,0], [0,0,0,1,1,1,1,0,0,0,0,0,0,0,0], [0,1,1,1,1,0,0,0,0,0,0,0,0,0,0],	a seven dimensional code of length 15

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Table A.1 – *Continued from previous page*

Notation	**Generating vectors**	**Description**
	[1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1]	
d_{16}	[0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1], [0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0], [0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0], [0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0], [0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0], [0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0], [1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0]	a seven dimensional code of length 16
e_{16}	[0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1], [0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0], [0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0], [0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0], [0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0], [0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0], [1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0], [0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1]	an eight dimensional code of length 16
d_{20}	[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1], [0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0], [0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0], [0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0], [0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0], [0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0], [0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0], [1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	a nine dimensional code of length 20

Next, we construct the representing codes for each genus (n, k, t) from the direct sum of the basic codes. The list of the genera and the corresponding codes from length 1 to 16 is in the following table.

Table A.2: The representing codes in each genus up to length 16

(n, k, t)	**Representing code**	(n, k, t)	**Representing code**
(1,0,odd)	t_1	(2,0,odd)	t_2
(3,0,odd)	t_3	(4,0,odd)	t_4
(4,1,even)	d_4	(5,0,odd)	t_5
(5,1,odd)	$t_1 \oplus d_4$	(6,0,odd)	t_6
(6,1,odd)	$t_2 \oplus d_4$	(6,2,odd)	d_6
(7,0,odd)	t_7	(7,1,odd)	$t_3 \oplus d_4$
(7,2,odd)	$t_1 \oplus d_6$	(7,3,odd)	e_7
(8,0,odd)	t_6	(8,1,odd)	$t_4 \oplus d_4$
(8,1,even)	h_8	(8,2,odd)	$t_2 \oplus d_6$
(8,2,even)	$d_4 \oplus d_4$	(8,3,odd)	$t_1 \oplus e_7$
(8,3,even)	d_8	(8,4,even)	e_8
(9,0,odd)	t_9	(9,1,odd)	$t_5 \oplus d_4$
(9,2,odd)	$t_3 \oplus d_6$	(9,3,odd)	$t_2 \oplus e_7$
(9,4,odd)	$t_1 \oplus e_8$	(10,0,odd)	t_{10}
(10,1,odd)	$t_6 \oplus d_4$	(10,2,odd)	$t_4 \oplus d_6$
(10,3,odd)	$t_3 \oplus e_7$	(10,4,odd)	$t_2 \oplus e_8$
(11,0,odd)	t_{11}	(11,1,odd)	$t_7 \oplus d_4$
(11,2,odd)	$t_5 \oplus d_6$	(11,3,odd)	$t_4 \oplus e_7$
(11,4,odd)	$t_3 \oplus e_8$	(12,0,odd)	t_{12}
(12,1,odd)	$t_8 \oplus d_4$	(12,1,even)	h_{12}
(12,2,odd)	$t_6 \oplus d_6$	(12,2,even)	$d_4 \oplus h_8$
(12,3,odd)	$t_5 \oplus e_7$	(12,3,even)	$d_4 \oplus d_4 \oplus d_4$
(12,4,odd)	$t_4 \oplus e_8$	(12,4,even)	$d_4 \oplus d_8$
(12,5,even)	$d_4 \oplus e_8$	(13,4,odd)	$t_5 \oplus e_8$

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Table A.2 – *Continued from previous page*

(n,k,t)	**Representing code**	(n,k,t)	**Representing code**
(13,5,odd)	$t_1 \oplus d_4 \oplus e_8$	(14,5,odd)	$t_2 \oplus d_{12}$
(14,6,odd)	$d_6 \oplus e_8$	(16,8,even)	e_{16}

Then we construct lattices from these codes. Each lattice will give us a lattice genus $\mathcal{G}(n,k,t)$ which we can compute the number of lattices in the corresponding genus directly using the command in Magma. And we get the result as in the table in table 4.3. However, when n is getting larger Magma cannot compute the number any more so we need to do in different way.

A.2 Indirect computation

We only need to compute the number of lattices in the highest dimension of the corresponding codes (possibly two highest dimensions in some cases). Since we cannot compute the number of lattices in each genus directly, we need to find different lattices in the genus one by one until we reach the sufficient number of lattices to prove that it is larger than the number of the corresponding codes. The process to find these lattices is as follow.

- First, we have to construct a list of as many lattices from the known codes as we can. Note that we can construct these codes using the basic codes. Then we choose one lattice from the list to begin with, say lattice L.

- Second, we have to find a suitable vector in L and use it to generate a non isometric lattice which belongs to the same genus as L one by one (this method is to find the neighboring lattice of L by using its vector).

  - Step 1 : we ask Magma to randomly choose a vector in L and then we test whether the vector is suitable or not to use the command 'Neighbour' in Magma.

If not we choose a new vector. We ask Magma to do this until we get the suitable one. Note that the suitable vector means its norm has to be divisible by the square of a prime number p that we use in the command 'Neighbour', i.e., if we use $p = 3$, then the norm of the vector has to be divisible by 9.

 - Step 2 : we ask Magma to generate a neighboring lattice, say L', of L from the vector we got from step 1.

 - Step 3 : we use the command 'IsIsometric' to test whether the lattice L' is isometric to each lattice in our list or not. If it is isometric, then we need to do from step 1 again until we get the non isometric lattice.

- Third, we add the non isomorphic lattice L' to our list of lattices and we repeat all the above steps again with possibly the same lattice L or we can choose another lattice from the list. Finally, we stop when we get enough number of lattices which should be at least one more than the number of the corresponding codes.

Therefore we need to compute all of the lattices in the following genera; $\mathcal{G}(18, 8, \text{odd})$, $\mathcal{G}(19, 8, \text{odd})$, $\mathcal{G}(20, 8, \text{odd})$, $\mathcal{G}(20, 9, \text{even})$,$\mathcal{G}(21, 9, \text{odd})$, $\mathcal{G}(22, 10, \text{odd})$, and $\mathcal{G}(23, 11, \text{odd})$

A.2.1 Lists of doubly even codes and the corresponding lattices

Some codes can be written as the direct sum of the basic codes as in table A.1. Some codes will be written explicitly with the generating vectors.

Table A.3: Lists of doubly even codes and the corresponding lattices

Genus	**Codes**	**Corresponding lattices**
(18,8,odd)	$t_2 \oplus e_{16}$	$L18_1$
	$e_8 \oplus d_{10}$	$L18_2$
	$t_2 \oplus e_8 \oplus e_8$	$L18_3$
	[1,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0,0,0],	
	[0,1,0,0,0,0,0,0,1,0,1,0,1,0,0,0,0,0],	

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Table A.3 – *Continued from previous page*

Genus	**Codes**	**Corresponding lattices**
	[0,0,1,0,0,0,0,0,1,0,0,1,1,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,1,1,1,0,1,1,1,0], [0,0,0,0,1,0,0,0,0,1,0,0,0,1,0,0,1,0], [0,0,0,0,0,1,0,0,0,1,1,1,1,1,1,1,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,0,1,1,0,1,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0]	$L18_4$
(19,8,odd)	[1,0,0,0,0,0,0,0,1,1,1,1,1,1,1,0,0,0,0], [0,1,0,0,0,0,0,0,1,1,1,1,1,1,0,1,0,0,0], [0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0,0], [0,0,0,1,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0], [0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,1,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,1,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0]	$L19_1$
	[1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0], [0,1,0,0,0,0,0,0,1,1,0,1,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,0,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,0,1,1,0,1,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0]	$L19_2$
	[1,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0,0,0,0], [0,1,0,0,0,0,0,0,1,0,1,0,1,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,0,1,1,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,1,1,1,0,1,1,1,0,0], [0,0,0,0,1,0,0,0,0,1,0,0,0,1,0,0,1,0,0], [0,0,0,0,0,1,0,0,0,1,1,1,1,1,1,1,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,0,1,1,0,1,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0]	$L19_3$
	[1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,0], [0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,1,1,0], [0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,0], [0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,1,1,0],	$L19_4$

Continued on next page

Table A.3 – *Continued from previous page*

Genus	**Codes**	**Corresponding lattices**
	[0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,1,1,0], [0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,1,1,0], [0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,1,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,0]	
	[1,0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0,0], [0,1,0,0,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0], [0,0,1,0,0,0,0,0,0,0,1,1,1,1,1,1,1,0,0], [0,0,0,1,0,0,0,0,0,0,1,0,0,1,0,0,0,1,0], [0,0,0,0,1,0,0,0,0,0,1,1,1,0,1,1,1,1,0], [0,0,0,0,0,1,0,0,0,0,0,1,0,1,0,0,0,1,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0,0,1,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,1,0]	$L19_5$
	[1,0,0,0,0,0,0,0,1,1,1,0,0,0,1,1,1,1,0], [0,1,0,0,0,0,0,0,1,0,0,1,1,1,1,0,1,1,0], [0,0,1,0,0,0,0,0,1,1,1,1,1,1,0,1,0,0,0], [0,0,0,1,0,0,0,0,0,1,0,1,1,1,1,0,1,1,0], [0,0,0,0,1,0,0,0,0,0,1,1,1,1,1,0,1,1,0], [0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,1,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0]	$L19_6$
	[1,0,0,0,0,0,0,0,1,0,1,1,1,1,1,0,1,0,0], [0,1,0,0,0,0,0,0,1,0,1,1,1,1,1,0,0,1,0], [0,0,1,0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,0], [0,0,0,1,0,0,0,0,0,1,1,0,1,1,0,1,1,1,0], [0,0,0,0,1,0,0,0,0,1,0,1,1,1,0,1,1,1,0], [0,0,0,0,0,1,0,0,0,1,1,1,0,0,0,0,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0]	$L19_7$
	$d_4 \oplus e_{15}$	$L19_8$
	$e_7 \oplus d_{12}$	$L19_9$
	$t_1 \oplus e_8 \oplus d_{10}$	$L19_{10}$
	$d_4 \oplus e_7 \oplus e_8$	$L19_{11}$
(20,8,odd)	$d_6 \oplus d_{14}$	$L20O_1$
	$d_{10} \oplus d_{10}$	$L20O_2$

Continued on next page

Table A.3 – *Continued from previous page*

Genus	**Codes**	**Corresponding lattices**
	$d_6 \oplus e_7 \oplus e_7$	$L20O_3$
	$t_2 \oplus e_8 \oplus d_{10}$	$L20O_4$
	$t_1 \oplus d_4 \oplus e_{15}$	$L20O_5$
	$t_1 \oplus e_7 \oplus d_{12}$	$L20O_6$
	[1,0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0,0,0], [0,1,0,0,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0], [0,0,1,0,0,0,0,0,0,0,1,1,1,1,1,1,1,0,0,0], [0,0,0,1,0,0,0,0,0,0,1,0,0,1,0,0,0,1,0,0], [0,0,0,0,1,0,0,0,0,0,1,1,1,0,1,1,1,1,0,0], [0,0,0,0,0,1,0,0,0,0,0,1,0,1,0,0,0,1,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0,0,1,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,1,0,0]	$L20O_7$
	[1,0,0,0,0,0,0,0,1,1,1,0,0,0,1,1,1,1,0,0], [0,1,0,0,0,0,0,0,1,0,0,1,1,1,1,0,1,1,0,0], [0,0,1,0,0,0,0,0,1,1,1,1,1,1,0,1,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,0,1,1,1,1,0,1,1,0,0], [0,0,0,0,1,0,0,0,0,0,1,1,1,1,1,0,1,1,0,0], [0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,1,0,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0]	$L20O_8$
	[1,0,0,0,0,0,0,0,1,0,1,1,1,1,1,0,1,0,0,0], [0,1,0,0,0,0,0,0,1,0,1,1,1,1,1,0,0,1,0,0], [0,0,1,0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,0,0], [0,0,0,1,0,0,0,0,0,1,1,0,1,1,0,1,1,1,0,0], [0,0,0,0,1,0,0,0,0,1,0,1,1,1,0,1,1,1,0,0], [0,0,0,0,0,1,0,0,0,1,1,1,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0]	$L20O_9$
	[1,0,0,0,0,0,0,0,1,1,1,1,1,1,1,0,1,1,1,1], [0,1,0,0,0,0,0,0,1,1,1,1,1,1,0,1,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,1,1,1,1,1], [0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,1,0,0,0,0],	$L20O_{10}$

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Table A.3 – *Continued from previous page*

Genus	Codes	Corresponding lattices
	[0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0]	
	[1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0], [0,1,0,0,0,0,0,0,1,1,0,1,0,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,0,1,1,0,1,0,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0]	$L20O_{11}$
	[1,0,0,0,0,0,0,0,1,1,1,1,1,1,1,0,0,0,0,0], [0,1,0,0,0,0,0,0,1,1,1,1,1,1,0,1,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,1,0,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,1,0,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0]	$L20O_{12}$
	[1,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0,0,0,0,0], [0,1,0,0,0,0,0,0,1,0,1,0,1,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,0,1,1,0,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,1,1,1,0,1,1,1,0,0,0], [0,0,0,0,1,0,0,0,0,1,0,0,0,1,0,0,1,0,0,0], [0,0,0,0,0,1,0,0,0,1,1,1,1,1,1,1,0,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,0,1,1,0,1,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0]	$L20O_{13}$
	[1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,0,0], [0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,1,1,0,0], [0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0], [0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,1,1,0,0], [0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,1,1,0,0], [0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,1,1,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,1,0,0],	$L20O_{14}$

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Table A.3 – *Continued from previous page*

Genus	Codes	Corresponding lattices
	[0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,0,0]	
	[1,0,0,0,0,0,0,0,1,1,1,1,1,1,1,0,1,1,1,1], [0,1,0,0,0,0,0,0,1,1,1,1,1,1,0,1,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,1,1,1,1,1], [0,0,0,1,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,1,1,1,1,1], [0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,1,0,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,1,1,1,1], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0]	$L20O_{15}$
	[1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,1,1,1,1], [0,1,0,0,0,0,0,0,1,1,0,1,0,0,0,0,1,1,1,1], [0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,0,1,1,0,1,0,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0]	$L20O_{16}$
	[1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,1,1,1,1], [0,1,0,0,0,0,0,0,1,0,1,1,0,0,0,0,1,1,1,1], [0,0,1,0,0,0,0,0,1,1,0,1,0,0,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0,1,1,1,1], [0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,0,1,1,0,1,1,1,1,1], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0]	$L20O_{17}$
	[0,1,0,0,0,0,0,0,1,1,0,1,0,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,0,1,1,0,1,0,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0], [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1]	$L20O_{18}$

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Table A.3 – *Continued from previous page*

Genus	Codes	Corresponding lattices
	[1,0,0,0,0,0,0,1,1,0,1,0,1,1,0,0,1,1,0,0], [0,1,0,0,0,0,0,0,1,0,0,1,1,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,1,1,0,1,0,1,0,1,0,1,1,0,0], [0,0,0,1,0,0,0,1,0,1,1,0,0,1,1,1,1,0,0,0], [0,0,0,0,1,0,0,1,1,0,1,0,1,0,0,1,1,1,0,0], [0,0,0,0,0,1,0,1,0,1,0,0,0,0,0,0,0,1,0,0], [0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,0,1,1,0,0], [0,0,0,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,1,1]	$L20O_{19}$
	[1,0,0,0,0,0,0,1,1,0,0,0,1,0,1,1,1,1,0,0], [0,1,0,0,0,0,0,0,1,0,0,1,1,0 0,0,0,0,0,0], [0,0,1,0,0,0,0,1,1,0,0,0,1,1,0,1,1,1,0,0], [0,0,0,1,0,0,0,1,1,0,0,0,1,1,1,1,0,1,0,0], [0,0,0,0,1,0,0,1,1,0,0,0,1,1,1,0,1,1,0,0], [0,0,0,0,0,1,0,1,1,0,0,0,1,1,1,1,1,0,0,0], [0,0,0,0,0,0,1,0,0,1,1,0,0,1,1,1,1,1,0,0], [0,0,0,0,0,0,0,0,0,1,0,0,0,1,1,1,1,1,1,1]	$L20O_{20}$
	[1,0,0,0,0,0,0,1,0,1,1,0,0,0,1,1,1,1,0,0], [0,1,0,0,0,0,0,0,1,0,0,1,0,1,1,1,1,1,0 0], [0,0,1,0,0,0,0,1,0,1,1,0,0,1,0,1,1,1,0,0], [0,0,0,1,0,0,0,1,0,1,1,0,0,1,1,1,0,1,0,0], [0,0,0,0,1,0,0,1,0,1,1,0,0,1,1,0,1,1,0,0], [0,0,0,0,0,1,0,1,1,0,0,0,0,0,0,0,0,1,0,0], [0,0,0,0,0,0,1,0,1,0,0,0,1,1,1,1,1,1,0,0], [0,0,0,0,0,0,0,0,1,0,0,0,0,1,1,1,1,1,1,1]	$L20O_{21}$
	[1,0,0,0,0,0,0,1,1,1,1,1,0,1,0,0,0,1,0,0], [0,1,0,0,0,0,0,1,0,0,0,1,1,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,1,1,1,1,1,0,0,1,0,0,1,0,0], [0,0,0,1,0,0,0,1,1,1,1,1,0,0,0,0,1,1,0,0], [0,0,0,0,1,0,0,1,1,1,1,1,0,0,0,1,0,1,0,0], [0,0,0,0,0,1,0,1,0,0,0,1,0,1,1,1,1,1,0,0], [0,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0,1,0,0], [0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,1,1]	$L20O_{22}$

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Table A.3 – *Continued from previous page*

Genus	Codes	Corresponding lattices
	[1,0,0,0,0,0,0,0,0,0,1,0,0,1,0,1,0,0,0], [0,1,0,0,0,0,0,1,0,0,1,0,0,0,0,0,1,0,0], [0,0,1,0,0,0,0,0,1,1,0,0,0,1,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,0,1,0,0,0,1,1,0,0,0], [0,0,0,0,1,0,0,0,0,0,1,0,0,1,1,0,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,0,0 0], [0,0,0,0,0,0,1,0,0,0,1,0,1,0,0,0,0,1,0,0], [0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,1]	$L20O_{23}$
	[1,0,0,0,0,0,0,1,1,1,0,1,1,1,1,0,0,0,0,0], [0,1,0,0,0,0,0,1,0,1,1,0,0,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,1,0,1,0,0,0,0,0,0,0], [0,0,0,0 1,0,0,0,0,1,1,0,0,1,0,0,0,0,0 0], [0,0,0,0,0,1,0,1,1,1,0,1,1,1,0,1,0,0,0,0], [0,0,0,0,0,0,1,1,0,1,0,1,1,1,1,1,0,0,0,0], [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1]	$L20O_{24}$
	[1,0,0,0,0,0,0,0,1,1,1,0,1,1,1,1,0,0,0,0], [0,1,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,0,1,1,1,0,0 0,0,0,0,0], [0,0,0,0,1,0,0,1,1,0,0,0,0,0,0,1,0,0,0,0], [0,0,0,0,0,1,0,1,0,0,0,0,0,1,0,1,0,0,0,0], [0,0,0,0,0,0,1,1,0,0,0,0,0,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1]	$L20O_{25}$
	[1,0,0,0,0,0,0,1,1,1,1,1,0,0,0,0,1,1,0,0], [0,1,0,0,0,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,1,0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0], [0,0,0,0,1,0 0,0,1,1,1,0,1,1,0,1,1,0,0,0], [0,0,0,0,0,1,0,1,0,0,0,1,0,0,0,0,0 0,1,0], [0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,1,1,0], [0,0,0,0,0,0,0,1,0,0,1,1,1,1,0,1,0,1,0,1]	$L20O_{26}$
	[1,0,0,0,0,0,0,1,1,1,1,0,0,1,1,1,0,0,0,0],	

Continued on next page

Table A.3 – *Continued from previous page*

Genus	**Codes**	**Corresponding lattices**
	[0,1,0,0,0,0,0,0,1,0,0,1,0,1,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,0,0,1,1,0,0,0,0,0,0], [0,0,0,1,0,0,0,1,1,0,1,0,0,1,1,1,0,1,0,0], [0,0,0,0,1,0,0,1,0,0,1,0,0,0,0,0,1,0,0,0], [0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,1,0,0,0], [0,0,0,0,0,0,1,1,1,1,1,0,0,1,0,1,0,1,0,0], [0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,1]	$L20O_{27}$
	[1,0,0,0,0,0,0,0,1,1,1,0,0,0,1,1,1,1,0,0], [0,1,0,0,0,0,0,0,0,1,0,1,0,0,1,0,0,0,0,0], [0,0,1,0,0,0,0,0,0,1,0,0,1,0,1,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,0,0,0,1,1,0,0,0,0,0], [0,0,0,0,1,0,0,1,0,0,1,0,0,0,0,0,0,1,0,0], [0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,0,1,0,0], [0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,1,1,0,0], [0,0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,0,1,1]	$L20O_{28}$
	[1,0,0,0,0,0,0,0,1,1,1,1,0,1,1,1,0,0,0,0], [0,1,0,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0], [0,0,0,0,1,0,0,1,1,0,0,0,0,0,1,0,0,0,0,0], [0,0,0,0,0,1,0,0,1,1,1,1,0,1,1,0,1,0,0,0], [0,0,0,0,0,0,1,0,1,0,1,1,0,1,1,1,1,0,0,0], [0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0]	$L20O_{29}$
	[1,0,0,0,0,0,0,0,1,1,1,0,0,1,1,0,1,1,0,0], [0,1,0,0,0,0,0,0,0,1,0,1,0,1,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,0,1,0,0,1,1,0,0,0,0,0,0], [0,0,0,1,0,0,0,1,0,0,1,0,0,0,1,0,0,0,0,0], [0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,1,0,1,0,0], [0,0,0,0,0,1,0,0,1,0,0,0,0,0,0,1,1,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,1,1,0,0], [0,0,0,0,0,0,0,1,1,1,0,0,0,1,1,0,1,1,1,0]	$L20O_{30}$
	[1,0,0,0,0,0,0,0,1,1,1,0,0,0,1,1,1,1,0,0], [0,1,0,0,0,0,0,0,0,1,0,1,0,0,1,0,0,0,0,0],	

Continued on next page

Table A.3 – *Continued from previous page*

Genus	**Codes**	**Corresponding lattices**
	[0,0,1,0,0,0,0,0,0,1,0,0,1,0,1,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,0,0,0,1,1,0,0,0,0,0], [0,0,0,0,1,0,0,1,0,0,1,0,0,0,0,0,0,1,0,0], [0,0,0,0,0,1 0,1,0,0,0,0,0,0,0,1,0,1,0,0], [0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,1,1,0,0], [0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,1,1,0]	$L20O_{31}$
	[1,0,0,0,0,0,0,1,0,1,1,0,0,0,1,1,1,1,0,0], [0,1,0,0,0,0,0,0,1,0,0,1,0,1,1,1,1,1,0,0], [0,0,1,0,0,0,0,1,1,1,1,0,0,0,1,1,1,0,0,0], [0,0 0,1,0,0,0,1,0,0,0,0,0,1,1,0,0,0,0,0], [0,0,0,0,1,0,0,1,0,0,0,0,0,1,0,1,0,0,0,0], [0,0,0,0,0,1,0,1,0,0,0,0,0,1,0,0,1,0,0,0], [0,0,0,0,0,0,1,0,1,0,0,0,1,1,1,1,1,1,0,0], [0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,1,1,0]	$L20O_{32}$
	[1,0,0,0,0,0,0,1,1,0,0,0,1,0,1,1,1,1,0,0], [0,1,0,0,0,0,0,0,1,0,0,1,1,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0], [0,0,0,1,0,0,0,1,0,0,0,0,0,1,1,0,0,0,0,0], [0,0,0,0,1,0,0,1,0,0,0,0,0,1,0,1,0,0,0,0], [0,0,0,0,0,1,0,1,0,0,0,0,0,1,0,0,1,0,0,0], [0,0,0,0,0,0,1,0,0,1,1,0,0,1,1,1,1,1,0,0], [0,0,0,0,0,0,0,1,0,0,0,1,1,0,1,1,1,1,1,0]	$L20O_{33}$
	[1,0,0,0,0,0,0,1,1,1,1,0,1,0,1,0,0,1,0,0], [0,1,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0,0,0], [0,0,1,0,0,0,0,1,1,1,0,0,1,0,1,0,0,1,1,0], [0,0,0,1,0,0,0,1,0,0,1,0,1,0,0,1,1,1,1,0], [0,0,0,0,1,0,0,1,0,0,1,1,1,0,0,0,1,1,1,0], [0,0,0,0,0,1,0,0,0,0,0,0,1,1,0,0,0,1,0,0], [0,0,0,0,0,0,1,1,0,0,1,1,1,0,0,1,0,1,1,0], [0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,1,1]	$L20O_{34}$
	[1,0,0,0,0,0,0,0,1,1,0,1,0,0,1,1,0,1,1,0], [0,1,0,0,0,0,0,0,0,0,1,1,1,0,1,1,1,0,1,0], [0,0,1,0,0,0,0,0,1,1,1,1,0,0,1,1,0,1,0,0],	

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Table A.3 – *Continued from previous page*

Genus	**Codes**	**Corresponding lattices**
	[0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,0], [0,0,0,0,1,0,0,0,1,0,0,0,0,0,1,0,1,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,1,1,0,0], [0,0,0,0,0,0,1,0,0,0,1,1,0,1,1,1,1,0,1,0], [0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,1,1]	$L20O_{35}$
	[1,0,0,0,0,0,0,1,1,0,0,1,0,0,1,1,1,1,0,0], [0,1,0,0,0,0,0,0,0,1,1,1,1,0,0,1,1,1,0,0], [0,0,1,0,0,0,0,1,0,1,1,0,0,0,1,1,1,0,1,0], [0,0,0,1,0,0,0,1,1,0,0,0,0,0,0,0,0,0,1,0], [0,0,0,0,1,0,0,1,0,1,1,0,0,0,1,0,1,1,1,0], [0,0,0,0,0,1,0,0,0,1,1,1,0,1,0,1,1,1,0,0], [0,0,0,0,0,0,1,1,0,1,1,0,0,0,1,1,0,1,1,0], [0,0,0,0,0,0,0,0,1,0,0,1,0,0,1,1,1,1,1,1]	$L20O_{36}$
	[1,0,0,0,0 0,0,0,0,0,0,1,1,0,1,0,0,0,0,0], [0,1,0,0,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0,0], [0,0,1,0,0,0,0,0,0,1,1,1,1,1,0,1,0,1,0,0], [0,0,0,1,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0], [0,0,0,0,1,0,0,0,0,1,1,1,1,1,0,0,1,1,0,0], [0,0,0,0,0,1,0,1,1,0,0,1,1,1,0,1,1,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,1,1,0,0], [0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,0,1,0]	$L20O_{37}$
	[1,0,0,0,0,0,0,1,0,0,1,0,0,1,1,1,1,1,0,0], [0,1,0,0,0,0,0,0,1,1,0,0,1,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,0,0,1,1,0,1,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,0,1,0,1,0,1,0,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,1,0,1,0,0,1,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,1,0,1,0,0,0,1,0,0], [0,0,0,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,1,1]	$L20O_{38}$
	[1,0,0,0,0,0,0,1,1,1,1,0,0,0,1,0,1,1,0,0], [0,1,0,0,0,0,0,0,0,0,0,1,0,1,0,0,0,0,1,0], [0,0,1,0,0,0,0,0,0,0,1,0,1,0,1,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,0,0,1,0,1,0,1,0,0,0,0],	$L20O_{39}$

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Table A.3 – *Continued from previous page*

Genus	**Codes**	**Corresponding lattices**
	[0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,0,1,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,0,1,0,1,0,0,1,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,0,1,0,1,0,0,1,0], [0,0,0,0,0,0,0,1,0,0,0,0,1,0,1,0,0,0,0,1]	
	[1,0,0,0,0,0,0,0,0,0,1,1,1,1,1,0,1,1,0,0], [0,1,0,0,0,0,0,1,1,1,0,1,1,1,1,0,0,0,0,0], [0,0,1,0,0,0,0,0,1,1,0,1,0,1,0,1,1,1,0,0], [0,0,0,1,0,0,0,0,1,1,0,1,0,0,1,1,1,1,0,0], [0,0,0,0,1,0,0,0,0,0,1,1,1,1,1,1,1,0,0,0], [0,0,0,0,0,1,0,1,0,0,0,1,0,0,0,0,1,0,0,0], [0,0,0,0,0,0,1,0,1,1,0,1,1,0,0,1,1,1,0,0], [0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,1,1]	$L20O_{40}$
	[1,0,0,0,0,0,0,1,1,1,1,0,0,1,1,1,0,0,0,0], [0,1,0,0,0,0,0,0,0,0,0,1,0,0,0,0,1,1,0,0], [0,0,1,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0,0,0], [0,0,0,1,0,0,0,1,0,0,0,0,1,0,1,0,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,1,1,1,1,1,1,1,0,0,0], [0,0,0,0,0,1,0,0,1,1,0,1,1,1,1,0,0,1,0,0], [0,0,0,0,0,0,1,0,1,1,0,0,1,1,1,0,1,1,0,0], [0,0,0,0,0,0,0,1,1,1,0,1,1,0,0,1,1,0,1,0]	$L20O_{41}$
	[1,0,0,0,0,0,0,1,1,0,0,0,1,0,1,1,1,1,0,0], [0,1,0,0,0,0,0,0,1,1,1,1,0,0,1,1,1,0,0,0], [0,0,1,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0,0,0], [0,0,0,1,0,0,0,1,0,1,1,0,0,1,1,0,1,1,0,0], [0,0,0,0,1,0,0,0,1,1,0,1,1,1,1,0,0,1,0,0], [0,0 0,0,0,1,0 0,1,1,0,1,1,1,0,0,1,1,0,0], [0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,1,0,1,0,0], [0,0,0,0,0,0,0,1,0,0,0,1,1,0,1,1,1,1,1,0]	$L20O_{42}$
	[1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,0,0], [0,1,0,0,0,0,0,0,1,0,1,1,0,1,1,1,0,1,0,0], [0,0,1,0,0,0,0,1,1,0,1,1,1,1,0,0,1,0,0,0], [0,0,0,1,0,0,0,0,1,0,0,0,0,0,1,0,1,0,0,0], [0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,1,1,0,0,0],	$L20O_{43}$

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Table A.3 – *Continued from previous page*

Genus	**Codes**	**Corresponding lattices**
	[0,0,0,0,0,1,0,0,0,1,1,0,0,0,1,1,1,1,1,0], [0,0,0,0,0,0,1,0,0,0,1,0,0,1,0,0,0,0,1,0], [0,0,0,0,0,0,0,0,0,1,0,0,0,1,1,1,1,1,1,1]	
	[1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,0,0], [0,1,0,0,0,0,0,0,1,0,1,1,0,1,1,1,0,1,0,0], [0,0,1,0,0,0,0,1,1,0,1,1,1,1,0,0,1,0,0,0], [0,0,0,1,0,0,0,0,1,0,0,0,0,0,1,0,1,0,0,0], [0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,1,1,0,0,0], [0,0,0,0,0,1,0,0,0,1,1,0,0,0,1,1,1,1,1,0], [0,0,0,0,0,0,1,0,0,0,1,0,0,1,0,0,0 0,1,0], [0,0,0,0,0,0,0,0,1,0,1,0,1,1,1,1,0,1,0,1]	$L20O_{44}$
	[1,0,0,0,0,0,0,1,1,1,1,1,1,0,1,1,1,1,1,0], [0,1,0,0,0,0,0,1,1,1,0,0,1,1,0,0,0,1,1,0], [0,0,1,0,0,0,0,1,1,1,1,0,1,1,0,0,0,0,1,0], [0,0,0,1,0,0,0,0,0,0,1,1,1,1,1,0,0,1,1,0], [0,0,0,0,1,0,0,0,0,0,1,1,1,1,0,1,0,1,1,0], [0,0,0,0,0,1,0,1,1,1,1,0,1,1,0,0,0,1,0,0], [0,0,0,0,0,0,1,0,0,0,1,1,1,1,0,0,1,1,1,0], [0,0,0,0,0,0,0,0,0,1,0,1,0,1,0,0,0,0,0,1]	$L20O_{45}$
	[1,0,0,0,0,0,0,1,0,1,0,0,0,1,1,1,1,0,1,0], [0,1,0,0,0,0,0,1,0,1,1,1,0,0,0,1,0,1,1,0], [0,0,1,0,0,0,0,1,1,0,1,1,0,0,1,0,1 0,1,0], [0,0,0,1,0,0,0,0,1,0,0,0,1,1,1,0,1,1,1,0], [0,0,0,0,1,0,0,1,0,0,0,0,0,0,1,0,0,1,0,0], [0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,1,1,0,0], [0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,1,0,0,1,0], [0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,1,1]	$L20O_{46}$
	[1,0,0,0,0,0,0,1,0,1,0,0,0,1,1,1,1,0,1,0], [0,1,0,0,0,0,0,1,0,1,1,1,0,0,0,1,0,1,1,0], [0,0,1,0,0,0,0,1,1,0,1,1,0,0,1,0,1,0,1,0], [0,0,0,1,0,0,0,0,1,0,0,0,1,1,1,0,1,1,1,0], [0,0,0,0,1,0,0,1,0,0,0,0,0,0,1,0,0,1,0,0], [0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,1,1,0,0],	$L20O_{47}$

Continued on next page

Table A.3 – *Continued from previous page*

Genus	Codes	Corresponding lattices
	[0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,1,0,0,1,0], [0,0,0,0,0,0,0,1,1,1,0,0,1,1,0,1,0,1,0,1]	
	[1,0,0,0,0,0,0,1,0,1,0,0,0,1,1,1,1,1,0,0], [0,1,0,0,0,0,0,0,1,0,1,1,1,1,0,0,0,1,1,0], [0,0,1,0,0,0,0,1,1,0,1,1,0,0,1,0,1,1,0,0], [0,0,0,1,0,0,0,1,0,1,0,0,0,0,1,1,1,1,1,0], [0,0,0,0,1,0,0,0,1,1,0,0,1,1,1,1,0,0,1,0], [0,0,0,0,0,1,0,0,1,0,0,0,0,0,0,1,0,1,0,0], [0,0,0,0,0,0,1,0,1,1,0,0,1,1,0,1,1,0,1,0], [0,0,0,0,0,0,0,1,0,0,1,0,1,1,1,0,1,0,1,1]	$L20O_{48}$
	[1,0,0,0,0,0,0,0,1,1,0,0,0,1,1,1,1,1,0,0], [0,1,0,0,0,0,0,1,0,1,1,1,0,0,0,1,1,1,0,0], [0,0,1,0,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0], [0,0,0,1,0,0,0,1,1,0,1,1,1,0,0,0,1,0,1,0], [0,0,0,0,1,0,0,1,0,0,0,0,0,0,1,0,1,0,0,0], [0,0,0,0,0,1,0,0,0,0,1,1,0,1,1,1,1,0,1,0], [0,0,0,0,0,0,1,0,0,0,1,1,0,1,1,0,1,1,1,0], [0,0,0,0,0,0,0,1,1,1,1,1,0,1,1,0,0,0,0,1]	$L20O_{49}$
	[1,0,0,0,0,0,0,1,1,0,0,0,0,1,1,0,1,1,1,0], [0,1,0,0 0,0,0,1,1,1,0,1,0,1,0,1,0,0,1,0], [0,0,1,0,0,0,0,1,1,0,1,1,0,0,1,1,1,0,0,0], [0,0,0,1,0,0,0,0,1,1,1,0,1,0,1,0,1,1,0,0], [0,0,0,0,1,0,0,1,0,1,1,0,0,1,1,1,0,1,0,0], [0,0,0,0,0,1,0,1,0,1,1,0,0,1,0,1,1,1,0,0], [0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,0,0,1,1,0], [0,0,0,0,0,0,0,0,1,0,0,0,0,1,1,1,1,1,1,1]	$L20O_{50}$
	[1,0,0,0,0,0,0,1,1,0,0,0,0,1,1,0,1,1,1,0], [0,1,0,0,0,0,0,1,1,1,0,1,0,1,0,1,0,0,1,0], [0,0,1,0,0,0,0,1,1,0,1,1,0,0,1,1,1,0,0,0], [0,0,0,1,0,0,0,0,1,1,1,0,1,0,1,0,1,1,0,0], [0,0,0 0,1,0,0,1,0,1,1,0,0,1,1,1,0,1,0,0], [0,0,0,0,0,1,0,1,0,1,1,0,0,1,0,1,1,1,0,0], [0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,0,0,1,1,0],	$L20O_{51}$

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Table A.3 – *Continued from previous page*

Genus	**Codes**	**Corresponding lattices**
	[0,0,0,0,0,0,0,1,0,1,0,0,1,0,1,1,1,0,1,1]	
	[1,0,0,0,0,0,0,1,0,1,1,0,1,1,1,0,0,1,0,0], [0,1,0,0,0,0,0,0,0,1,1,1,1,1,0,0,1,0,1,0], [0,0,1,0,0,0,0,1,1,0,0,1,1,0,1,1,0,1,0,0], [0,0,0,1,0,0,0,0,0,0,1,0,1,0,1,1,1,1,1,0], [0,0,0,0,1,0,0,1,1,0,0,0,1,1,1,0,1,0,1,0], [0,0,0,0,0,1,0,0,1,1,0,0,1,1,0,1,1,1,0,0], [0,0,0,0,0,0,1,1,1,1,1,0,1,0,0,1,0,0,1,0], [0,0,0,0,0,0,0,1,0,0,1,1,0,1,1,1,0,0,1,1]	$L20O_{52}$
	[1,0,0,0,0,0,0,1,1,1,1,1,1,0,1,1,1,1,1,0], [0,1,0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,0,0,0,1,1,0,1,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,0,0,1,1,0,0,1,0,0,0], [0,0 0,0,0,1,0,0,0,0,0,0,1,1,0,0,0,1,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0,0,0,1,0], [0,0,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0,0,1]	$L20O_{53}$
(20,8,even)	d_{20}	$L20E_1$
	$d_4 \oplus d_8 \oplus e_8$	$L20E_2$
	$d_8 \oplus d_{12}$	$L20E_3$
	[1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,1,1,1,1], [0,1,0,0,0,0,0,0,1,1,0,1,0,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,0,1,1,0,1,0,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0]	$L20E_4$
	[1,0,0,0,0,0,0,0,1,1,1,1,1,1,1,0,1,1,1,1], [0,1,0,0,0,0,0,0,1,1,1,1,1,1,0,1,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,1,0,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,1,0,0,0,0],	$L20E_5$

Continued on next page

Table A.3 – *Continued from previous page*

Genus	Codes	Corresponding lattices
	[0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0]	
	[1,0,0,0,0,0,0,0,1,1,1,1,1,1,1,0,0,0,0,0], [0,1,0,0,0,0,0,0,1,1,1,1,1,1,0,1,1,1,1,1], [0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0,0,0], [0,0,0,1,0,0,0,0,0,1,0,0,0,0,1,1,1,1,1,1], [0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,1,0,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,1,1,1,1,1], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0]	$L20E_6$
	[1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,1,1,1,1], [0,1,0,0,0,0,0,0,1,1,0,1,0,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0,1,1,1,1], [0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0], [0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,0,1,1,1,1], [0,0,0,0,0,1,0,0,0,0,0,0,1,1,0,1,0,0,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0]	$L20E_7$
	[1,0,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0], [0,1,0,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,0,0,1,0,1,0,1,0,1,0,0,0,0,0], [1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,1,1,1], [0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,0,0,0], [1,1,0,0,1,1,0,0,1,1,0,0,1,1,0,0,0,0,0,0], [1,1,1,1,0,0,0,0,1,1,1,1,0,0,0,0,1,1,1,1], [0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0]	$L20E_8$
	[1,0,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0], [0,1,0,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,0,0,1,0,1,0,1,0,1,0,0,0,0,0], [1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,1,1,1], [0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,0,0,0], [1,1,0,0,1,1,0,0,1,1,0,0,1,1,0,0,0,0,0,0], [0,0,1,0,1,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0],	$L20E_9$

Continued on next page

Table A.3 – *Continued from previous page*

Genus	Codes	Corresponding lattices
	[1,1,1,1,1,1,1,1,0,0,0,0,1,1,1,1,0,0,0,0]	
	[1,0,0,0,0,0,0,0,0,0,1,1,1,1,0,1,0,1,1,0], [0,1,0,0,0,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0], [0,0,1,0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0], [0,0,0,0,1,0 0,0,1,1,1,0,1,1,0,1,1,0,0,0], [0,0,0,0,0,1,0,1,0,0,0,1,0,0,0,0,0,1,0,0], [0,0,0,0,0,0,1,0,1,1,0,0,1,1,0,1,1,0,1,0], [0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,1,1,1,1]	$L20E_{10}$
	[1,0,0,0,0,0,0,1,1,1,1,1,0,0,1,1,0,0,0,0], [0,1,0,0,0,0,0,0,1,0,0,0,1,0,1,0,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0,0], [0,0,0,1,0,0,0,1,1,0,1,1,0,0,1,1,0,0,1,0], [0,0,0,0,1,0,0,1,0,0,0,0,0,0,0,1,1,0,0,0], [0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,0,1,0,0], [0,0,0,0,0,0,1,1,1,1,0,0,0,0,1,0,1,1,1,0], [0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,1,1,1,1,1]	$L20E_{11}$
	[1,0,0,0,0,0,1,1,1,1,1,0,1,1,0,0,0,0,0,0], [0,1,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0], [0,0,0,1,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0], [0,0,0,0,1,0,1,0,1,0 0,0,0,0,1,0,0,0,0,0], [0,0,0,0,0,1,1,0,0,0,0,0,0,1,1,0,0,0,0,0], [0,0,0,0,0,0,0,0,1,0,0,0,0,1,1,1,0,0,0,0], [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1]	$L20E_{12}$
	[1,0,0,0,0,0,0,0,1,1,1,0,0,1,1,0,0,1,1,0], [0,1,0,0,0,0,0,0,0,1,0,0,1,1,0,0,0,0,0,0], [0,0,1,0,0,0,0,0,1,0,1,1,0,0,0,1,1,1,1,0], [0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,0], [0,0,0,0,1,0,0,0,1,0,0,0,0,0,1,0,1,0,0,0], [0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,1,1,0,0], [0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,0,1,0], [0,0,0,0,0,0,0,0,0,1,1,1,0,1,1,1,1,0,0,1]	$L20E_{13}$

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Table A.3 – *Continued from previous page*

Genus	Codes	Corresponding lattices
	[1,0,0,0,0,0,0,1,1,1,1,1,0,1,0,0,0,1,0,0], [0,1,0,0,0,0,0,0,1,0,0,0,1,1,0,0,0,0,0,0], [0,0,1,0,0,0,0,1,0,0,1,0,0,0,0,0,0,0,1,0], [0,0,0,1,0,0,0,1,0,1,0,1,0,0,0,1,1,1,1,0], [0,0,0,0,1,0,0,1,1,0,0,0,0,1,1,0,1,1,1,0], [0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,0,0,1,0,0], [0,0,0,0,0,0,1,1,1,0,0,0,0,1,1,1,0,1,1,0], [0,0,0,0,0,0,0,0,1,1,1,0,0,1,0,1,1,0,1,1]	$L20E_{14}$
	[1,0,0,0,0,0,0,1,1,0,0,1,0,0,1,1,1,1,0,0], [0,1,0,0,0,0,0,0,0,1,1,1,1,0,0,1,1,1,0,0], [0,0,1,0,0,0,0,1,0,1,1,0,0,0,1,1,1,0,1,0], [0,0,0,1,0,0,0,1,1,0,0,0,0,0,0,0,0,0,1,0], [0,0,0,0,1,0,0,1,0,1,1,0,0,0,1,0,1,1,1,0], [0,0,0,0,0,1,0,0,0,1,1,1,0,1,0,1,1,1,0,0], [0,0,0,0,0,0,1,1,0,1,1,0,0,0,1,1,0,1,1,0], [0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,1]	$L20E_{15}$
	[1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,0,0], [0,1,0,0,0,0,0,0,1,0,1,1,0,1,1,1,0,1,0,0], [0,0,1,0,0,0,0,1,1,0,1,1,1,1,0,0,1,0,0,0], [0,0,0,1,0,0,0,0,1,0,0,0,0,0,1,0,1,0,0,0], [0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,1,1,0,0,0], [0,0 0,0,0,1,0,0,0,1,1,0,0,0,1,1,1,1,1,0], [0,0,0,0,0,0,1,0,0,0,1,0,0,1,0,0,0,0,1,0], [0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,1,1]	$L20E_{16}$
	[1,0,0,0,0,0,0,1,0,1,0,0,0,1,1,1,1,0,1,0], [0,1,0,0,0,0,0,1,0,1,1,1,0,0,0,1,0,1,1,0], [0,0,1,0,0,0,0,1,1,0,1,1,0,0,1,0,1,0,1,0], [0,0,0,1,0,0,0,0,1,0,0,0,1,1,1,0,1,1,1,0], [0,0,0,0,1,0,0,1,0,0,0,0,0,0,1,0,0,1,0,0], [0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,1,1,0,0], [0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,1,0,0,1,0], [0,0,0,0,0,0,0,0,0,1,1,1,0,1,1,0,1,1,0,1]	$L20E_{17}$

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Table A.3 – *Continued from previous page*

Genus	**Codes**	**Corresponding lattices**
	[1,0,0,0,0,0,0,1,0,1,0,0,1,1,0,1,0,1,1,0], [0,1,0,0,0,0,0,1,0,0,0,1,1,1,1,0,1,0,1,0], [0,0,1,0,0,0,0,1,1,0,1,1,1,0,0,0,0,1,1,0], [0,0,0,1,0,0,0,1,0,1,1,0,1,0,1,0,1,1,0,0], [0,0,0,0,1,0,0,0,1,1,1,0,1,1,0,0,1,0,1,0], [0,0,0,0,0,1,0,0,1,1,1,0,1,1,1,0,0,0,1,0], [0,0,0,0,0,0,1,0,1,0,0,0,1,0,1,1,1,1,1,0], [0,0,0,0,0,0,0,1,1,1,1,1,0,1,1,1,1,1,1,1]	$L20E_{18}$
	[1,0,0,0,0,0,0,1,1,0,0,0,0,1,1,0,1,1,1,0], [0,1,0,0,0,0,0,1,1,1,0,1,0,1,0,1,0,0,1,0], [0,0,1,0,0,0,0,1,1,0,1,1,0,0,1,1,1,0,0,0], [0,0,0,1,0,0,0,0,1,1,1,0,1,0,1,0,1,1,0,0], [0,0,0,0,1,0,0,1,0,1,1,0,0,1,1,1,0,1,0,0], [0,0,0,0,0,1,0,1,0,1,1,0,0,1,0,1,1,1,0,0], [0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,0,0,1,1,0], [0,0,0,0,0,0,0,0,1,0,1,1,0,1,1,1,1,0,0,1]	$L20E_{19}$
(20,9,even)	$d_4 \oplus e_8 \oplus e_8$	$L20^9_1$
	$d_4 \oplus e_{16}$	$L20^9_2$
	d_{20}	$L20^9_3$
	$e_8 \oplus d_{12}$	$L20^9_4$
(21,9,odd)	[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,0], [0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0], [0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0], [0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0,0], [0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0], [0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0], [0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0], [1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	$L21_0$
	$t_1 \oplus e_8 \oplus d_{12}$	$L21_1$
	$t_1 \oplus d_4 \oplus e_8 \oplus e_8$	$L21_2$
	$t_1 \oplus d_4 \oplus e_{16}$	$L21_3$
	$e_7 \oplus e_7 \oplus e_7$	$L21_4$
	$d_6 \oplus e_{15}$	$L21_5$
	$d_6 \oplus e_7 \oplus e_8$	$L21_6$

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Table A.3 – *Continued from previous page*

Genus	Codes	Corresponding lattices
(22,10,odd)	$e_7 \oplus e_{15}$	$L22_1$
	$d_6 \oplus e_{16}$	$L22_2$
	$d_6 \oplus e_8 \oplus e_8$	$L22_3$
	$e_8 \oplus d_{14}$	$L22_4$
(23,11,odd)	$e_7 \oplus e_{16}$	$L23_0$
	$e_8 \oplus e_{15}$	$L23_1$
	$e_7 \oplus e_8 \oplus e_8$	$L23_2$

A.2.2 Lists of vectors and corresponding lattices using command Neighbour(L,v,3)

Table A.4: Lists of vectors and corresponding lattices using command **Neighbour(L,v,3)**

Genus	Vectors	Corresponding lattices
(18,8,odd)	v_5 = (0 -5 1 1 0 0 0 0 1 0 0 0 2 0 0 0 0 0) from $L18_1$	$L18_5$
	v_6 = (0 -3 0 1 0 1 1 0 0 1 0 1 0 0 0 0 1 0) from $L18_2$	$L18_6$
	v_7 = (0 0 0 1 0 0 0 1 0 0 1 0 0 1 1 0 1 0) from $L18_2$	$L18_7$
	v_8 = (1 0 -3 0 0 0 1 2 0 0 0 0 0 0 1 0 1 1) from $L18_2$	$L18_8$
	v_9 = (0 0 0 1 1 0 0 0 0 1 0 2 1 0 0 1 0 0) from $L18_2$	$L18_9$
	v_{10} = (-5 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0) from $L18_4$	$L18_{10}$
(19,8,odd)	v_{12} = (-1 0 0 0 0 0 0 1 0 0 1 0 0 1 0 1 0 0 1) from $L19_{11}$	$L19_{12}$
	v_{13} = (-3 0 0 0 1 0 0 2 0 1 0 0 0 0 1 0 0 1 1) from $L19_{11}$	$L19_{13}$
	v_{14} = (0 -5 0 0 2 0 0 0 0 1 0 0 0 1 0 0 2 0 0) from $L19_{11}$	$L19_{14}$
	v_{15} = (-3 0 0 0 0 0 1 0 2 1 0 0 0 0 0 1 0 1 1) from $L19_{10}$	$L19_{15}$
	v_{16} = (0 0 0 1 0 1 0 1 1 1 0 0 2 0 0 0 0 0 0) from $L19_{10}$	$L19_{16}$
	v_{17} = (-3 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 2 1 1) from $L19_5$	$L19_{17}$
	v_{18} = (-3 2 0 0 0 1 0 0 0 1 1 0 2 0 0 0 0 0 0) from $L19_5$	$L19_{18}$
	v_{19} = (-6 1 1 0 0 0 1 1 0 0 1 0 0 1 0 0 1 0 0) from $L19_5$	$L19_{19}$
	v_{20} = (0 0 1 0 0 0 0 0 0 1 0 0 0 3 0 1 0 1 0) from $L19_4$	$L19_{20}$
(20,8,odd)	v_{54} = (0 -1 0 0 0 0 1 0 0 0 0 0 0 1 1 0 0 0 2 1) from $L20O_{10}$	$L20O_{54}$
	v_{55} = (-3 0 0 0 1 1 2 0 1 0 0 0 0 0 0 0 2 0 1 0) from $L20O_{10}$	$L20O_{55}$
	v_{56} = (0 0 0 0 0 1 0 0 1 0 1 1 0 1 0 0 1 1 1 1) from $L20O_{10}$	$L20O_{56}$

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Table A.4 – *Continued from previous page*

lattices		
	v_{57} = (-1 0 1 0 2 0 0 0 0 0 3 0 0 1 0 0 0 0 0 0) from $L20O_{15}$	$L20O_{57}$
	v_{58} = (-6 0 1 0 0 0 0 0 0 2 1 2 0 0 0 0 1 1 0 0) from $L20O_{15}$	$L20O_{58}$
	v_{59} = (1 -3 0 0 0 0 1 3 0 0 0 0 0 0 1 0 0 1 1 0) from $L20O_{16}$	$L20O_{59}$
	v_{60} = (0 2 0 0 0 0 1 0 0 0 0 0 2 0 0 1 0 1 1 0) from $L20O_{12}$	$L20O_{60}$
	v_{61} = (0 0 0 1 0 0 1 0 2 0 0 0 0 1 1 1 1 0 0 0) from $L20O_{13}$	$L20O_{61}$
	v_{62} = (0 -6 0 2 0 0 0 0 0 0 0 1 2 0 0 0 0 2 1 0) from $L20O_{13}$	$L20O_{62}$
	v_{63} = (-3 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 2 0 2 2) from $L20O_{13}$	$L20O_{63}$
	v_{64} = (0 0 0 1 0 0 1 0 1 1 0 1 0 0 1 0 1 0 1 0) from $L20O_7$	$L20O_{64}$
	v_{65} = (0 0 1 0 0 4 0 1 0 0 0 0 0 1 0 0 0 0 0 1) from $L20O_7$	$L20O_{65}$
	v_{66} = (-1 0 0 1 0 0 1 0 1 0 0 0 0 0 1 3 0 1 0 0) from $L20O_9$	$L20O_{66}$
	v_{67} = (1 -6 0 1 0 1 0 1 0 0 1 0 0 0 0 1 0 0 1 1) from $L20O_9$	$L20O_{67}$
(20,9,even)	v_5 = (-3 0 1 1 0 0 0 1 0 1 0 0 0 0 0 3 2 1 0 0) from $L20^9_4$	$L20^9_5$
	v_6 = (-3 0 0 0 0 0 2 0 2 1 1 0 0 1 1 0 1 0 0 0) from $L20^9_1$	$L20^9_6$
	v_7 = (0 0 0 0 0 2 0 1 0 2 1 0 0 0 0 1 0 1 1 0) from $L20^9_1$	$L20^9_7$
	v_8 = (2 -3 0 0 0 0 2 0 1 0 0 1 0 1 2 0 0 1 0 0) from $L20^9_2$	$L20^9_8$
	v_9 = (1 1 0 0 0 0 1 0 0 0 0 0 0 0 0 1 1 1 1 0) from $L20^9_2$	$L20^9_9$
	v_{10} = (-3 1 0 0 0 0 0 0 0 1 0 1 1 0 1 0 0 1 2 2) from $L20^9_2$	$L20^9_{10}$
	v_{11} = (-6 1 0 1 0 1 0 0 2 0 2 0 2 0 0 0 0 1 0 2) from $L20^9_2$	$L20^9_{11}$
	v_{12} = (-3 0 0 0 1 0 0 1 0 0 1 0 1 1 0 0 0 3 1 0) from $L20^9_4$	$L20^9_{12}$
(21,9,odd)	v_7 = (-6 0 0 1 0 0 1 0 0 0 0 0 0 0 2 1 1 0 0 1 1) from $L21_0$	$L21_7$
	v_8 = (-6 0 0 1 0 0 2 0 0 0 0 1 0 1 1 0 1 0 0 0 1) from $L21_0$	$L21_8$
	v_9 = (-3 0 0 0 0 0 0 2 0 2 0 1 0 1 0 0 1 1 1 0 0) from $L21_0$	$L21_9$
	v_{10} = (-6 0 2 0 0 0 1 1 1 0 1 0 0 1 0 0 0 0 0 1 0) from $L21_1$	$L21_{10}$
	v_{11} = (0 0 -3 1 0 1 0 1 0 1 0 0 0 0 2 0 0 2 0 0 0) from $L21_1$	$L21_{11}$
	v_{12} = (0 0 -3 1 0 0 0 0 0 4 0 0 0 1 0 0 1 0 0 0 1) from $L21_1$	$L21_{12}$
	v_{13} = (-6 0 0 0 0 1 0 2 0 0 1 0 1 0 0 0 1 1 0 1 1) from $L21_1$	$L21_{13}$
	v_{14} = (-6 0 0 1 0 1 0 1 0 0 1 1 0 0 1 0 1 0 1 0 1) from $L21_1$	$L21_{14}$
	v_{15} = (0 0 1 0 0 0 0 0 0 1 1 0 1 0 1 1 1 0 1 0 1) from $L21_1$	$L21_{15}$
	v_{16} = (0 0 0 0 0 0 0 0 -2 1 0 2 0 0 0 1 1 0 0 1 0) from $L21_2$	$L21_{16}$
	v_{17} = (-3 0 0 0 0 1 0 0 0 0 1 0 0 0 1 0 1 4 0 0 0) from $L21_2$	$L21_{17}$
	v_{18} = (-6 0 0 0 0 1 1 0 0 0 2 0 0 1 0 0 1 0 2 0 0) from $L21_2$	$L21_{18}$
	v_{19} = (0 0 0 0 1 0 1 0 0 1 0 1 0 0 0 0 0 2 0 2 0) from $L21_2$	$L21_{19}$
	v_{20} = (-6 0 0 0 0 1 0 0 1 0 1 1 0 1 1 1 1 0 0 0 0) from $L21_2$	$L21_{20}$
	v_{21} = (0 0 1 0 0 1 1 0 0 0 0 1 0 1 0 1 1 1 0 0 0) from $L21_2$	$L21_{21}$
	v_{22} = (-6 0 0 0 0 0 1 0 0 1 3 0 0 0 1 1 0 0 1 0 0) from $L21_2$	$L21_{22}$
	v_{23} = (-6 0 0 1 1 0 0 0 1 2 1 1 0 0 0 0 0 1 1 0 0) from $L21_2$	$L21_{23}$
	v_{24} = (-3 0 0 0 0 0 1 0 2 1 0 0 1 1 0 0 1 1 0 1 0) from $L21_2$	$L21_{24}$

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Table A.4 – *Continued from previous page*

lattices		
	v_{25} = (0 0 -3 2 0 1 0 0 0 0 0 0 1 1 0 0 1 0 1 1 2) from $L21_2$	$L21_{25}$
	v_{26} = (-3 1 0 0 0 1 2 0 0 1 0 0 1 2 1 0 0 1 0 0 0) from $L21_2$	$L21_{26}$
	v_{27} = (-6 0 0 2 0 0 0 1 0 1 0 1 0 0 0 0 0 1 0 1 0) from $L21_4$	$L21_{27}$
	v_{28} = (0 0 0 0 0 1 1 1 0 0 0 1 1 0 0 1 1 0 0 0 0) from $L21_4$	$L21_{28}$
	v_{29} = (-6 0 0 0 1 1 0 0 1 0 2 0 1 0 0 0 1 0 0 0 1) from $L21_4$	$L21_{29}$
	v_{30} = (-3 0 0 0 0 0 0 0 1 1 0 1 0 1 1 0 1 1 0 0 0) from $L21_5$	$L21_{30}$
	v_{31} = (-6 1 1 0 1 0 0 0 0 1 0 0 0 0 1 1 0 1 0 0 0) from $L21_5$	$L21_{31}$
	v_{32} = (0 0 0 0 0 0 1 1 0 0 1 0 1 0 1 0 0 1 0 1 0) from $L21_5$	$L21_{32}$
	v_{33} = (0 0 0 0 2 0 1 0 1 0 0 0 1 1 0 0 0 0 1 0 0) from $L21_5$	$L21_{33}$
	v_{34} = (0 0 0 1 0 0 1 0 1 0 0 0 0 1 0 0 1 1 1 0 0) from $L21_5$	$L21_{34}$
	v_{35} = (-3 0 1 1 1 0 0 0 0 0 0 0 1 1 0 1 1 0 0 0 1) from $L21_5$	$L21_{35}$
	v_{36} = (0 0 0 0 0 0 0 2 1 0 0 1 1 0 0 1 1 0 1 0 1) from $L21_5$	$L21_{36}$
	v_{37} = (1 0 1 0 0 0 0 0 0 0 1 2 0 0 0 1 0 1 0 0 1) from $L21_5$	$L21_{37}$
	v_{38} = (1 0 0 0 1 0 0 0 0 1 0 1 0 0 1 1 0 0 2 0 0) from $L21_5$	$L21_{38}$
(22,10,odd)	v_5 = (0 0 0 1 0 0 0 0 1 1 0 1 1 0 0 0 0 0 0 1 0 1) from $L22_4$	$L22_5$
	v_6 = (-3 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 1 2 0 0 1) from $L22_4$	$L22_6$
	v_7 = (0 0 1 0 1 0 0 1 0 2 1 0 0 1 1 0 0 0 0 0 1 0) from $L22_4$	$L22_7$
	v_8 = (-6 0 0 1 1 0 1 1 1 0 0 1 2 0 0 0 0 0 1 0 0 0) from $L22_4$	$L22_8$
	v_9 = (-6 0 1 0 0 0 0 1 1 0 0 2 0 0 0 0 0 0 0 1 1 0) from $L22_1$	$L22_9$
	v_{10} = (-6 0 0 1 0 0 0 0 1 0 1 1 0 0 1 0 0 0 2 0 0 0) from $L22_1$	$L22_{10}$
	v_{11} = (0 0 0 0 0 0 0 1 0 0 0 0 0 1 1 1 2 1 0 1 0 0) from $L22_1$	$L22_{11}$
	v_{12} = (1 0 0 0 1 0 0 0 1 0 0 0 4 1 0 0 0 0 0 0 0 0) from $L22_1$	$L22_{12}$
	v_{13} = (2 0 -6 0 0 0 0 2 0 1 0 0 0 0 1 0 1 0 1 0 0 0) from $L22_1$	$L22_{13}$
	v_{14} = (-6 0 0 0 0 1 0 1 0 0 2 0 1 0 0 1 0 0 0 1 1 0) from $L22_1$	$L22_{14}$
	v_{15} = (-6 0 0 0 0 0 1 1 0 1 1 1 0 0 1 0 0 1 0 0 0 1) from $L22_1$	$L22_{15}$
	v_{16} = (0 1 -3 0 1 0 1 0 1 0 0 1 1 0 0 0 0 0 0 2 0 0) from $L22_1$	$L22_{16}$
	v_{17} = (-3 0 0 0 0 1 0 0 1 0 1 0 1 1 0 1 1 0 1 0 0 0) from $L22_1$	$L22_{17}$
	v_{18} = (1 -6 0 0 0 0 0 0 1 0 0 0 1 0 0 1 3 1 0 0 0 0) from $L22_1$	$L22_{18}$
	v_{19} = (-2 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 2 1 0 1) from $L22_2$	$L22_{19}$
	v_{20} = (0 0 0 1 0 0 0 1 1 0 1 0 0 1 0 0 0 0 1 0 0 1) from $L22_2$	$L22_{20}$
	v_{21} = (-6 0 0 0 0 1 0 1 0 0 0 0 0 2 1 0 0 1 0 0 0 2) from $L22_2$	$L22_{21}$
	v_{22} = (-6 0 1 0 0 0 0 0 1 0 0 0 0 2 0 0 2 1 0 1 0 0) from $L22_2$	$L22_{22}$
	v_{23} = (0 1 1 0 0 1 0 0 2 0 0 0 0 1 0 0 0 0 0 1 1 0) from $L22_2$	$L22_{23}$
	v_{24} = (0 -5 0 0 0 0 1 0 0 1 0 0 0 1 0 0 1 1 1 1 0 0) from $L22_2$	$L22_{24}$
	v_{25} = (0 2 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 1 0 1 1 1) from $L22_2$	$L22_{25}$
	v_{26} = (0 0 0 0 0 0 0 0 0 1 -6 0 0 1 1 0 2 0 0 1 0 1) from $L22_3$	$L22_{26}$

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Table A.4 – *Continued from previous page*

lattices		
	v_{27} = (-3 0 0 1 0 4 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 1) from $L22_4$	$L22_{27}$
(23,11,odd)	v_3 = (-6 1 0 1 0 0 0 0 0 1 0 0 1 1 0 0 0 0 0 2 0 0 0) from $L23_0$	$L23_3$
	v_4 = (0 0 0 0 0 0 0 0 1 0 1 0 0 1 1 0 1 2 1 0 0 0 0) from $L23_0$	$L23_4$
	v_5 = (-3 0 0 1 0 0 0 0 0 2 0 0 0 1 0 2 0 0 1 1 0 0 0) from $L23_0$	$L23_5$
	v_6 = (-6 1 0 0 0 0 0 0 1 0 0 0 3 0 0 0 0 1 0 0 2 0 0) from $L23_0$	$L23_6$
	v_7 = (-3 0 1 0 0 1 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1 0) from $L23_0$	$L23_7$
	v_8 = (-6 1 0 0 0 0 1 0 0 0 0 0 0 0 1 2 2 0 0 0 1 0 0) from $L23_0$	$L23_8$
	v_9 = (-3 0 0 0 2 0 0 0 1 1 0 1 2 0 0 0 0 1 0 0 0 2 0) from $L23_1$	$L23_9$
	v_{10} = (2 -3 0 0 0 0 1 2 0 1 0 0 0 0 0 0 0 2 0 1 1 0 0) from $L23_1$	$L23_{10}$
	v_{11} = (1 0 0 0 4 0 0 1 1 0 0 1 0 0 1 0 0 0 0 0 1 1 0) from $L23_1$	$L23_{11}$

A.2.3 Table comparing the number of lattices from code and lattices from computation

The following table contains the numbers of codes, lattices constructed from known codes, and lattices constructed by computation.

genus	codes	lattices from known codes	lattices from computation	total lattices
$\mathcal{G}(18, 8, \text{odd})$	9	4	6	10
$\mathcal{G}(19, 8, \text{odd})$	19	11	9	20
$\mathcal{G}(20, 8, \text{odd})$	84	53	14	67*
$\mathcal{G}(20, 8, \text{even})$		19		
$\mathcal{G}(20, 9, \text{even})$	10	4	8	12
$\mathcal{G}(21, 9, \text{odd})$	38	7	32	39
$\mathcal{G}(22, 10, \text{odd})$	25	4	23	27
$\mathcal{G}(23, 11, \text{odd})$	11	3	9	12

From the table, we can see that the number in the last column is larger than the number in the second column which means the least number of lattices in each genus is larger than the number of codes. Hence these lattices genera are non code type.

Note that (*) in the genera $\mathcal{G}(20, 8, \text{odd})$ and $\mathcal{G}(20, 8, \text{even})$, we cannot find the exact number of codes in each genus but we know that $\mathcal{G}(20, 8, \text{even})$ is non code type because the genus $\mathcal{G}(20, 9, \text{even})$ is non code type. We know only the total number of codes (20,8) which is 84. And we found 53 codes of odd type and 19 codes of even type. Then we found 14 lattices of odd type by computation so we have 67 lattices of odd type. But when we combine 67 odd type lattices with 19 even type lattices we have a total of 86 lattices which exceeds the number of codes. So we have at least two lattices of odd type that cannot be constructed from codes, and hence $\mathcal{G}(20, 8, \text{odd})$ is non code type.

Appendix B

Tables for the classification of genera of VOAs for small MTCs with central charges at most 16

Table B.1: The S-matrices, the components of the irreducible representations, and the corresponding canonical basis vectors of the representation corresponding to small MTCs

No.	$\mathcal{C}$	n	c (mod 8)	S	$\rho_1 \oplus ... \oplus \rho_s$	Basis vectors
1	t_m	1	8	(1)	$\underline{1}$	$[1]$
2	qs_2	2	1	$\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$	$\underline{2}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
3	$\overline{qs_2}$	2	7	$\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$	$\underline{2}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
4	$Lee - Yang$	2	14/5	$\begin{pmatrix} \sqrt{\frac{2}{5+\sqrt{5}}} & \frac{1+\sqrt{5}}{\sqrt{2(5+\sqrt{5})}} \\ \frac{1+\sqrt{5}}{\sqrt{2(5+\sqrt{5})}} & -\sqrt{\frac{2}{5+\sqrt{5}}} \end{pmatrix}$	$\underline{2}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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Table B.1 – *Continued from previous page*

No.	$\mathcal{C}$	n	c (mod 8)	S	$\rho_1 \oplus ... \oplus \rho_s$	Basis vectors
5	$\overline{Lee-Yang}$	2	26/5	$\begin{pmatrix} \sqrt{\frac{2}{5+\sqrt{5}}} & \frac{1+\sqrt{5}}{\sqrt{2(5+\sqrt{5})}} \\ \frac{1+\sqrt{5}}{\sqrt{2(5+\sqrt{5})}} & -\sqrt{\frac{2}{5+\sqrt{5}}} \end{pmatrix}$	$\underline{2}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
6	qs_3	3	2	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 \\ 1 & \omega+\omega^2 \end{pmatrix}$ where $\omega = e^{2\pi i/3}$	$\underline{1} \oplus \underline{2}$	$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$
7	$\overline{qs_3}$	3	6	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 \\ 1 & \omega+\omega^2 \end{pmatrix}$ where $\omega = e^{2\pi i/3}$	$\underline{1} \oplus \underline{2}$	$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$
8	$Ising1$	3	1/2	$\frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$	$\underline{3}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
9	$\overline{Ising1}$	3	15/2	$\frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$	$\underline{3}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
10	$Ising2$	3	3/2	$\frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$	$\underline{3}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
11	$\overline{Ising2}$	3	13/2	$\frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$	$\underline{3}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
12	$Ising3$	3	5/2	$\frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$	$\underline{3}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
13	$\overline{Ising3}$	3	11/2	$\frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$	$\underline{3}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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Table B.1 – *Continued from previous page*

No.	$\mathcal{C}$	n	c (mod 8)	S	$\rho_1 \oplus ... \oplus \rho_s$	Basis vectors
14	$Ising4$	3	7/2	$\frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$	$\underline{3}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
15	$\overline{Ising4}$	3	9/2	$\frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$	$\underline{3}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
16	$3fieldsx$	3	8/7	$\frac{1}{D}\begin{pmatrix} 1 & d & d^2-1 \\ d & -d^2+1 & 1 \\ d^2-1 & 1 & -d \end{pmatrix}$ where $d = 2cos(\frac{\pi}{7}), D = \frac{\sqrt{7}}{2sin(\frac{\pi}{7})}$	$\underline{3}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
17	$\overline{3fieldsx}$	3	48/7	$\frac{1}{D}\begin{pmatrix} 1 & d & d^2-1 \\ d & -d^2+1 & 1 \\ d^2-1 & 1 & -d \end{pmatrix}$ where $d = 2cos(\frac{\pi}{7}), D = \frac{\sqrt{7}}{2sin(\frac{\pi}{7})}$	$\underline{3}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
18	qs_4	4	1	$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \\ 1 & -1 & i & -i \end{pmatrix}$	$\underline{1} \oplus \underline{3}$	$\begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
19	$\overline{qs_4}$	4	7	$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \\ 1 & -1 & i & -i \end{pmatrix}$	$\underline{1} \oplus \underline{3}$	$\begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
20	qn_4	4	5	$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \\ 1 & -1 & i & -i \end{pmatrix}$	$\underline{1} \oplus \underline{3}$	$\begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
21	$\overline{qn_4}$	4	3		$\underline{1} \oplus \underline{3}$	$\begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}$,

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No.	$\mathcal{C}$	n	c (mod 8)	S	$\rho_1 \oplus ... \oplus \rho_s$	Basis vectors
				$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \\ 1 & -1 & i & -i \end{pmatrix}$		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
22	qu_2	4	8	$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$	$\underline{1} \oplus \underline{1} \oplus \underline{2}$	$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
23	qv_2	4	4	$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$	$\underline{1} \oplus \underline{1} \oplus \underline{2}$	$\begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$
24	$qs_2 \otimes qs_2$	4	2	$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$	$\underline{1} \oplus \underline{3}$	$\begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
25	$\overline{qs_2} \otimes \overline{qs_2}$	4	6	$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$	$\underline{1} \oplus \underline{3}$	$\begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
26	$qs_2 \otimes \overline{qs_2}$	4	8	$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$	$\underline{1} \oplus \underline{3}$	$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
27	$qs_2 \otimes LY$	4	19/5	$\frac{1}{D}\begin{pmatrix} 1 & d & 1 & d \\ d & -1 & d & -1 \\ 1 & d & -1 & -d \\ d & -1 & -d & 1 \end{pmatrix}$ where $D = \sqrt{5+\sqrt{5}}, d = \frac{1+\sqrt{5}}{2}$	$\underline{4}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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No.	$\mathcal{C}$	n	$c \pmod 8$	S	$\rho_1 \oplus ... \oplus \rho_s$	Basis vectors
28	$\overline{qs_2} \otimes LY$	4	9/5	$\frac{1}{D}\begin{pmatrix} 1 & d & 1 & d \\ d & -1 & d & -1 \\ 1 & d & -1 & -d \\ d & -1 & -d & 1 \end{pmatrix}$ where $D = \sqrt{5+\sqrt{5}}, d = \frac{1+\sqrt{5}}{2}$	$\underline{4}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
29	$qs_2 \otimes \overline{LY}$	4	31/5	$\frac{1}{D}\begin{pmatrix} 1 & d & 1 & d \\ d & -1 & d & -1 \\ 1 & d & -1 & -d \\ d & -1 & -d & 1 \end{pmatrix}$ where $D = \sqrt{5+\sqrt{5}}, d = \frac{1+\sqrt{5}}{2}$	$\underline{4}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
30	$\overline{qs_2} \otimes \overline{LY}$	4	21/5	$\frac{1}{D}\begin{pmatrix} 1 & d & 1 & d \\ d & -1 & d & -1 \\ 1 & d & -1 & -d \\ d & -1 & -d & 1 \end{pmatrix}$ where $D = \sqrt{5+\sqrt{5}}, d = \frac{1+\sqrt{5}}{2}$	$\underline{4}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
31	$LY \otimes LY$	4	28/5	$\left(\frac{1}{\sqrt{2+d}}\right)^2 \begin{pmatrix} 1 & d & d & d^2 \\ d & -1 & d^2 & -d \\ d & d^2 & -1 & -d \\ d^2 & -d & -d & 1 \end{pmatrix}$ where $d = \frac{1+\sqrt{5}}{2}$	$\underline{1} \oplus \underline{3}$	$\begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
32	$\overline{LY} \otimes \overline{LY}$	4	12/5	$\left(\frac{1}{\sqrt{2+d}}\right)^2 \begin{pmatrix} 1 & d & d & d^2 \\ d & -1 & d^2 & -d \\ d & d^2 & -1 & -d \\ d^2 & -d & -d & 1 \end{pmatrix}$ where $d = \frac{1+\sqrt{5}}{2}$	$\underline{1} \oplus \underline{3}$	$\begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
33	$LY \otimes \overline{LY}$	4	8	$\left(\frac{1}{\sqrt{2+d}}\right)^2 \begin{pmatrix} 1 & d & d & d^2 \\ d & -1 & d^2 & -d \\ d & d^2 & -1 & -d \\ d^2 & -d & -d & 1 \end{pmatrix}$ where $d = \frac{1+\sqrt{5}}{2}$	$\underline{1} \oplus \underline{3}$	$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

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No.	$\mathcal{C}$	n	c (mod 8)	S	$\rho_1 \oplus \ldots \oplus \rho_s$	Basis vectors
34	$4fieldsx$	4	10/3	$\frac{1}{D}\begin{pmatrix} 1 & d & d^2-1 & d+1 \\ d & -d-1 & d^2-1 & -1 \\ d^2 & d^2-1 & 0 & -d^2+1 \\ d+1 & -1 & -d^2+1 & d \end{pmatrix}$ where $D = \frac{3}{2sin(\frac{\pi}{9})}, d = 2cos(\frac{\pi}{9})$	$\underline{4}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
35	$\overline{4fieldsx}$	4	14/3	$\frac{1}{D}\begin{pmatrix} 1 & d & d^2-1 & d+1 \\ d & -d-1 & d^2-1 & -1 \\ d^2 & d^2-1 & 0 & -d^2+1 \\ d+1 & -1 & -d^2+1 & d \end{pmatrix}$ where $D = \frac{3}{2sin(\frac{\pi}{9})}, d = 2cos(\frac{\pi}{9})$	$\underline{4}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Table B.2: The exponent matrices and the characteristic matrices of the contributing irreducible representations of the small MTCs

No.	$\mathcal{C}$	n	c	Λ_i	$\mathcal{X}_i$
1	t_m	1	8	$\Lambda_1 = (\frac{2}{3})$	$\mathcal{X}_1 = (248)$
			16	$\Lambda_1 = (\frac{1}{3})$	$\mathcal{X}_1 = (496)$
2	qs_2	2	1	$\Lambda_1 = \text{Diag}(\frac{23}{24}, \frac{5}{24})$	$\mathcal{X}_1 = \begin{pmatrix} 3 & 26752 \\ 2 & -247 \end{pmatrix}$
			9	$\Lambda_1 = \text{Diag}(\frac{5}{8}, -\frac{1}{8})$	$\mathcal{X}_1 = \begin{pmatrix} 251 & 26752 \\ 2 & 1 \end{pmatrix}$
3	$\overline{qs_2}$	2	7	$\Lambda_1 = \text{Diag}(\frac{17}{24}, \frac{11}{24})$	$\mathcal{X}_1 = \begin{pmatrix} 133 & 1248 \\ 56 & -377 \end{pmatrix}$
			15	$\Lambda_1 = \text{Diag}(\frac{3}{8}, \frac{1}{8})$	$\mathcal{X}_1 = \begin{pmatrix} 381 & 1248 \\ 56 & -129 \end{pmatrix}$
4	$Lee-Yang$	2	14/5	$\Lambda_1 = \text{Diag}(\frac{53}{60}, \frac{17}{60})$	$\mathcal{X}_1 = \begin{pmatrix} 14 & 12857 \\ 7 & -258 \end{pmatrix}$
			54/5	$\Lambda_1 = \text{Diag}(\frac{11}{20}, -\frac{1}{20})$	$\mathcal{X}_1 = \begin{pmatrix} 262 & 12857 \\ 7 & -10 \end{pmatrix}$

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Table B.2 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Λ_i	$\mathcal{X}_i$
5	$\overline{Lee-Yang}$	2	26/5	$\Lambda_1=\mathrm{Diag}(\frac{47}{60},\frac{23}{60})$	$\mathcal{X}_1=\begin{pmatrix}52 & 3774\\ 26 & -296\end{pmatrix}$
			66/5	$\Lambda_1=\mathrm{Diag}(\frac{9}{20},\frac{1}{20})$	$\mathcal{X}_1=\begin{pmatrix}300 & 3774\\ 26 & -48\end{pmatrix}$
6	qs_3	3	2	$\Lambda_2=\mathrm{Diag}(\frac{11}{12},\frac{1}{4})$	$\mathcal{X}_2=\begin{pmatrix}8 & 78732\\ 1 & -252\end{pmatrix}$
			10	$\Lambda_2=\mathrm{Diag}(\frac{7}{12},-\frac{1}{12})$	$\mathcal{X}_2=\begin{pmatrix}256 & 78732\\ 1 & -4\end{pmatrix}$
7	$\overline{qs_3}$	3	6	$\Lambda_2=\mathrm{Diag}(\frac{3}{4},\frac{5}{12})$	$\mathcal{X}_2=\begin{pmatrix}78 & 91854\\ 1 & -322\end{pmatrix}$
			14	$\Lambda_2=\mathrm{Diag}(\frac{5}{12},\frac{1}{12})$	$\mathcal{X}_2=\begin{pmatrix}326 & 91854\\ 1 & -74\end{pmatrix}$
8	$Ising1$	3	1/2	$\Lambda_1=\mathrm{Diag}(\frac{47}{48},\frac{23}{48},\frac{1}{24})$	$\mathcal{X}_1=\begin{pmatrix}0 & 2325 & 94208\\ 1 & 275 & -4096\\ 1 & -25 & -23\end{pmatrix}$
			17/2	$\Lambda_1=\mathrm{Diag}(\frac{11}{16},\frac{3}{16},\frac{5}{8})$	$\mathcal{X}_1=\begin{pmatrix}136 & 5125 & 112\\ 17 & 123 & -16\\ 256 & -10496 & -7\end{pmatrix}$
9	$\overline{Ising1}$	3	15/2	$\Lambda_1=\mathrm{Diag}(\frac{11}{16},\frac{3}{16},\frac{5}{8})$	$\mathcal{X}_1=\begin{pmatrix}105 & 5083 & 288\\ 15 & 156 & -32\\ 128 & -4992 & -9\end{pmatrix}$
			31/2	$\Lambda_1=\mathrm{Diag}(\frac{17}{48},\frac{41}{48},\frac{7}{24})$	$\mathcal{X}_1=\begin{pmatrix}248 & 7 & 512\\ 3875 & 21 & -8704\\ 248 & -8 & -17\end{pmatrix}$
10	$Ising2$	3	3/2	$\Lambda_1=\mathrm{Diag}(\frac{15}{16},\frac{7}{16},\frac{1}{8})$	$\mathcal{X}_1=\begin{pmatrix}3 & 2871 & 43008\\ 3 & 270 & -2048\\ 2 & -54 & -21\end{pmatrix}$

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Table B.2 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Λ_i	$\mathcal{X}_i$
			19/2	$\Lambda_1 = \mathrm{Diag}(\frac{29}{48}, \frac{5}{48}, \frac{19}{24})$	$\mathcal{X}_1 = \begin{pmatrix} 171 & 5031 & 40 \\ 19 & 86 & -8 \\ 512 & -22016 & -5 \end{pmatrix}$
11	$\overline{Ising2}$	3	13/2	$\Lambda_1 = \mathrm{Diag}(\frac{35}{48}, \frac{11}{48}, \frac{13}{24})$	$\mathcal{X}_1 = \begin{pmatrix} 78 & 4921 & 704 \\ 13 & 185 & -64 \\ 64 & -2368 & -11 \end{pmatrix}$
			29/2	$\Lambda_1 = \mathrm{Diag}(\frac{19}{48}, \frac{43}{48}, \frac{5}{24})$	$\mathcal{X}_1 = \begin{pmatrix} 261 & 5 & 1024 \\ 3393 & 10 & -19456 \\ 116 & -4 & -19 \end{pmatrix}$
12	$Ising3$	3	5/2	$\Lambda_1 = \mathrm{Diag}(\frac{43}{48}, \frac{19}{48}, \frac{5}{24})$	$\mathcal{X}_1 = \begin{pmatrix} 10 & 3893 & 19456 \\ 5 & 261 & -1024 \\ 4 & -116 & -19 \end{pmatrix}$
			21/2	$\Lambda_1 = \mathrm{Diag}(\frac{9}{16}, \frac{1}{16}, \frac{7}{8})$	$\mathcal{X}_1 = \begin{pmatrix} 210 & 4785 & 12 \\ 21 & 45 & -4 \\ 1024 & -46080 & -3 \end{pmatrix}$
13	$\overline{Ising3}$	3	11/2	$\Lambda_1 = \mathrm{Diag}(\frac{37}{48}, \frac{13}{48}, \frac{11}{24})$	$\mathcal{X}_1 = \begin{pmatrix} 55 & 4655 & 1664 \\ 11 & 210 & -128 \\ 32 & -1120 & -13 \end{pmatrix}$
			27/2	$\Lambda_1 = \mathrm{Diag}(\frac{7}{16}, \frac{15}{16}, \frac{1}{8})$	$\mathcal{X}_1 = \begin{pmatrix} 270 & 3 & 2048 \\ 2871 & 3 & -43008 \\ 54 & -2 & -21 \end{pmatrix}$
14	$Ising4$	3	7/2	$\Lambda_1 = \mathrm{Diag}(\frac{41}{48}, \frac{17}{48}, \frac{7}{24})$	$\mathcal{X}_1 = \begin{pmatrix} 21 & 3875 & 8704 \\ 7 & 248 & -512 \\ 8 & -248 & -17 \end{pmatrix}$
			23/2	$\Lambda_1 = \mathrm{Diag}(\frac{25}{48}, \frac{1}{48}, \frac{23}{24})$	$\mathcal{X}_1 = \begin{pmatrix} 253 & 4371 & 2 \\ 23 & 0 & -2 \\ 2048 & -96256 & -1 \end{pmatrix}$

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Table B.2 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Λ_i	$\mathcal{X}_i$
15	$\overline{Ising4}$	3	9/2	$\Lambda_1 = \mathrm{Diag}(\frac{13}{16}, \frac{5}{16}, \frac{3}{8})$	$\mathcal{X}_1 = \begin{pmatrix} 36 & 4301 & 3840 \\ 9 & 231 & -256 \\ 16 & -258 & -15 \end{pmatrix}$
			25/2	$\Lambda_1 = \mathrm{Diag}(\frac{23}{48}, \frac{47}{48}, \frac{1}{24})$	$\mathcal{X}_1 = \begin{pmatrix} 275 & 1 & 4096 \\ 2325 & 0 & -94208 \\ 25 & -1 & -23 \end{pmatrix}$
16	$3fieldsx$	3	8/7	$\Lambda_1 = \mathrm{Diag}(-\frac{1}{21}, \frac{17}{21}, \frac{5}{21})$	$\mathcal{X}_1 = \begin{pmatrix} 14 & 5 & 11 \\ 50922 & -37 & 4797 \\ 782 & 17 & -217 \end{pmatrix}$
			64/7	$\Lambda_1 = \mathrm{Diag}(\frac{13}{21}, \frac{10}{21}, -\frac{2}{21})$	$\mathcal{X}_1 = \begin{pmatrix} 136 & 627 & 22990 \\ 117 & -374 & 3510 \\ 3 & 2 & -2 \end{pmatrix}$
17	$\overline{3fieldsx}$	3	48/7	$\Lambda_1 = \mathrm{Diag}(\frac{5}{7}, -\frac{1}{7}, \frac{3}{7})$	$\mathcal{X}_1 = \begin{pmatrix} 78 & 45954 & 1702 \\ 1 & 3 & 1 \\ 55 & 2925 & -321 \end{pmatrix}$
			104/7	$\Lambda_1 = \mathrm{Diag}(\frac{8}{21}, \frac{11}{21}, \frac{2}{21})$	$\mathcal{X}_1 = \begin{pmatrix} 188 & 138 & 1564 \\ 725 & -344 & 1972 \\ 44 & 11 & -84 \end{pmatrix}$
18	qs_4	4	1	$\Lambda_2 = \mathrm{Diag}(\frac{23}{24}, \frac{11}{24}, \frac{1}{12})$	$\mathcal{X}_2 = \begin{pmatrix} 1 & 2600 & 90112 \\ 2 & 273 & -4096 \\ 1 & -26 & -22 \end{pmatrix}$
			9	$\Lambda_2 = \mathrm{Diag}(\frac{5}{8}, \frac{1}{8}, \frac{3}{4})$	$\mathcal{X}_2 = \begin{pmatrix} 153 & 5096 & 96 \\ 18 & 105 & -16 \\ 256 & -10752 & -6 \end{pmatrix}$
19	$\overline{qs_4}$	4	7	$\Lambda_2 = \mathrm{Diag}(\frac{17}{24}, \frac{5}{24}, \frac{7}{12})$	$\mathcal{X}_2 = \begin{pmatrix} 91 & 5016 & 640 \\ 14 & 171 & -64 \\ 64 & -2432 & -10 \end{pmatrix}$

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Table B.2 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Λ_i	$\mathcal{X}_i$
			15	$\Lambda_2 = \mathrm{Diag}(\frac{3}{8}, \frac{7}{8}, \frac{1}{4})$	$\mathcal{X}_2 = \begin{pmatrix} 255 & 6 & 1024 \\ 3640 & 15 & -18432 \\ 120 & -4 & -18 \end{pmatrix}$
20	qn_4	4	5	$\Lambda_2 = \mathrm{Diag}(\frac{19}{24}, \frac{7}{24}, \frac{5}{12})$	$\mathcal{X}_2 = \begin{pmatrix} 45 & 4488 & 3584 \\ 10 & 221 & -256 \\ 16 & -544 & -14 \end{pmatrix}$
			13	$\Lambda_2 = \mathrm{Diag}(\frac{11}{24}, \frac{23}{24}, \frac{1}{12})$	$\mathcal{X}_2 = \begin{pmatrix} 273 & 2 & 4096 \\ 2600 & 1 & -90112 \\ 26 & -1 & -22 \end{pmatrix}$
21	$\overline{qn_4}$	4	3	$\Lambda_2 = \mathrm{Diag}(\frac{7}{8}, \frac{3}{8}, \frac{1}{4})$	$\mathcal{X}_2 = \begin{pmatrix} 15 & 3640 & 18432 \\ 6 & 255 & -1024 \\ 4 & -120 & -18 \end{pmatrix}$
			11	$\Lambda_2 = \mathrm{Diag}(\frac{13}{24}, \frac{1}{24}, \frac{11}{12})$	$\mathcal{X}_2 = \begin{pmatrix} 231 & 4600 & 8 \\ 22 & 23 & -4 \\ 1024 & -47104 & -2 \end{pmatrix}$
22	qu_2	4	8	$\Lambda_1 = (\frac{2}{3})$	$\mathcal{X}_1 = (248)$
				$\Lambda_3 = \mathrm{Diag}(\frac{2}{3}, \frac{1}{6})$	$\mathcal{X}_3 = \begin{pmatrix} -136 & 5120 \\ 48 & 140 \end{pmatrix}$
			16	$\Lambda_1 = (\frac{1}{3})$	$\mathcal{X}_1 = (496)$
				$\Lambda_3 = \mathrm{Diag}(\frac{1}{3}, \frac{5}{6})$	$\mathcal{X}_3 = \begin{pmatrix} -272 & 32 \\ 3072 & 28 \end{pmatrix}$
23	qv_2	4	4	$\Lambda_3 = \mathrm{Diag}(\frac{5}{6}, \frac{1}{3})$	$\mathcal{X}_3 = \begin{pmatrix} 28 & 12288 \\ 8 & -272 \end{pmatrix}$
			12	$\Lambda_3 = \mathrm{Diag}(\frac{1}{2}, 0)$	$\mathcal{X}_3 = \begin{pmatrix} 276 & 12288 \\ 8 & -24 \end{pmatrix}$
24	$qs_2 \otimes qs_2$	4	2	$\Lambda_2 = \mathrm{Diag}(\frac{11}{12}, \frac{1}{6}, \frac{5}{12})$	$\mathcal{X}_2 = \begin{pmatrix} 6 & 40960 & 3136 \\ 2 & -20 & -56 \\ 4 & -2048 & 266 \end{pmatrix}$

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Table B.2 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Λ_i	$\mathcal{X}_i$
			10	$\Lambda_2 = \mathrm{Diag}(\frac{7}{12}, \frac{5}{6}, \frac{1}{12})$	$\mathcal{X}_2 = \begin{pmatrix} 190 & 32 & 4928 \\ 512 & -4 & -22528 \\ 20 & -8 & 66 \end{pmatrix}$
25	$\overline{qs_2} \otimes \overline{qs_2}$	4	6	$\Lambda_2 = \mathrm{Diag}(\frac{3}{4}, \frac{1}{2}, \frac{1}{4})$	$\mathcal{X}_2 = \begin{pmatrix} 66 & 1536 & 4800 \\ 32 & -12 & -1152 \\ 12 & -128 & 198 \end{pmatrix}$
			14	$\Lambda_2 = \mathrm{Diag}(\frac{5}{12}, \frac{1}{6}, \frac{11}{12})$	$\mathcal{X}_2 = \begin{pmatrix} 266 & 2048 & 4 \\ 56 & -20 & -2 \\ 3136 & -40960 & 6 \end{pmatrix}$
26	$qs_2 \otimes \overline{qs_2}$	4	8	$\Lambda_1 = (\frac{2}{3})$	$\mathcal{X}_1 = (248)$
				$\Lambda_2 = \mathrm{Diag}(\frac{2}{3}, -\frac{1}{12}, \frac{5}{12})$	$\mathcal{X}_2 = \begin{pmatrix} 24 & 40960 & 2048 \\ 2 & -2 & 4 \\ 56 & 3136 & -262 \end{pmatrix}$
			16	$\Lambda_1 = (\frac{1}{3})$	$\mathcal{X}_1 = (496)$
				$\Lambda_2 = \mathrm{Diag}(\frac{1}{3}, \frac{1}{12}, \frac{7}{12})$	$\mathcal{X}_2 = \begin{pmatrix} 16 & 1536 & 128 \\ 32 & -62 & 12 \\ 1152 & 4800 & -194 \end{pmatrix}$
27	$qs_2 \otimes LY$	4	19/5	$\Lambda_1 = \mathrm{Diag}(\frac{101}{120}, \frac{29}{120}, \frac{11}{120}, \frac{59}{120})$	$\mathcal{X}_1 = \begin{pmatrix} 17 & 9945 & 16560 & 1456 \\ 7 & -143 & 392 & -56 \\ 2 & 52 & -51 & -26 \\ 14 & -884 & -2990 & 185 \end{pmatrix}$
			59/5	$\Lambda_1 = \mathrm{Diag}(\frac{61}{120}, -\frac{11}{120}, \frac{91}{120}, \frac{19}{120})$	$\mathcal{X}_1 = \begin{pmatrix} 193 & 8073 & 36 & 2392 \\ 3 & 1 & 2 & -4 \\ 592 & 47840 & -33 & -11063 \\ 40 & -208 & -13 & 95 \end{pmatrix}$

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Table B.2 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Λ_i	$\mathcal{X}_i$
28	$\overline{qs_2} \otimes LY$	4	9/5	$\Lambda_1 = \mathrm{Diag}(\frac{37}{40}, \frac{13}{40}, \frac{27}{40}, \frac{3}{40})$	$\mathcal{X}_1 = \begin{pmatrix} 3 & 8073 & 208 & 47840 \\ 3 & -189 & 40 & -592 \\ 4 & 2392 & -91 & -11063 \\ 2 & -36 & -13 & 37 \end{pmatrix}$
			49/5	$\Lambda_1 = \mathrm{Diag}(\frac{71}{120}, -\frac{1}{120}, \frac{41}{120}, \frac{89}{120})$	$\mathcal{X}_1 = \begin{pmatrix} 147 & 9945 & 884 & 52 \\ 7 & -13 & 14 & -2 \\ 56 & 1456 & -181 & -26 \\ 392 & -16560 & -2990 & 55 \end{pmatrix}$
29	$qs_2 \otimes \overline{LY}$	4	31/5	$\Lambda_1 = \mathrm{Diag}(\frac{89}{120}, \frac{41}{120}, -\frac{1}{120}, \frac{71}{120})$	$\mathcal{X}_1 = \begin{pmatrix} 55 & 2990 & 16560 & 392 \\ 26 & -181 & 1456 & -56 \\ 2 & 14 & -13 & -7 \\ 52 & -884 & -9945 & 147 \end{pmatrix}$
			71/5	$\Lambda_1 = \mathrm{Diag}(\frac{49}{120}, \frac{1}{120}, \frac{79}{120}, \frac{31}{120})$	$\mathcal{X}_1 = \begin{pmatrix} 211 & 2346 & 46 & 714 \\ 14 & -17 & 6 & -14 \\ 792 & 14280 & -95 & -3366 \\ 120 & -408 & -34 & 157 \end{pmatrix}$
30	$\overline{qs_2} \otimes \overline{LY}$	4	21/5	$\Lambda_1 = \mathrm{Diag}(\frac{33}{40}, \frac{17}{40}, \frac{23}{40}, \frac{7}{40})$	$\mathcal{X}_1 = \begin{pmatrix} 21 & 2346 & 408 & 14280 \\ 14 & -207 & 120 & -792 \\ 14 & 714 & -153 & -3366 \\ 6 & -46 & -34 & 99 \end{pmatrix}$
			61/5	$\Lambda_1 = \mathrm{Diag}(\frac{59}{120}, \frac{11}{120}, \frac{29}{120}, \frac{101}{120})$	$\mathcal{X}_1 = \begin{pmatrix} 185 & 2990 & 884 & 14 \\ 26 & -51 & 52 & -2 \\ 56 & 392 & -143 & -7 \\ 1456 & -16560 & -9945 & 17 \end{pmatrix}$
31	$LY \otimes LY$	4	28/5	$\Lambda_2 = \mathrm{Diag}(\frac{23}{30}, \frac{1}{6}, \frac{17}{30})$	$\mathcal{X}_2 = \begin{pmatrix} 28 & 16250 & 676 \\ 7 & 120 & -26 \\ 49 & -5000 & 104 \end{pmatrix}$

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Table B.2 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Λ_i	$\mathcal{X}_i$
			68/5	$\Lambda_2 = \mathrm{Diag}(\frac{13}{30}, \frac{5}{6}, \frac{7}{30})$	$\mathcal{X}_2 = \begin{pmatrix} 136 & 20 & 1196 \\ 1700 & 24 & -7475 \\ 119 & -10 & 92 \end{pmatrix}$
32	$LY \otimes \overline{LY}$	4	8	$\Lambda_1 = (\frac{2}{3})$	$\mathcal{X}_1 = (248)$
				$\Lambda_2 = \mathrm{Diag}(\frac{2}{3}, -\frac{4}{15}, \frac{1}{15})$	$\mathcal{X}_2 = \begin{pmatrix} -116 & 2500 & 8125 \\ 52 & -100 & 676 \\ 14 & 49 & -24 \end{pmatrix}$
			16	$\Lambda_1 = (\frac{1}{3})$	$\mathcal{X}_1 = (496)$
				$\Lambda_2 = \mathrm{Diag}(\frac{1}{3}, -\frac{1}{15}, \frac{11}{15})$	$\mathcal{X}_2 = \begin{pmatrix} -218 & 1275 & 25 \\ 10 & 1 & 3 \\ 4590 & 42483 & -23 \end{pmatrix}$
33	$\overline{LY} \otimes \overline{LY}$	4	12/5	$\Lambda_2 = \mathrm{Diag}(\frac{9}{10}, \frac{1}{2}, \frac{1}{10})$	$\mathcal{X}_2 = \begin{pmatrix} 3 & 2550 & 42483 \\ 5 & 222 & -2295 \\ 3 & -50 & 27 \end{pmatrix}$
			52/5	$\Lambda_2 = \mathrm{Diag}(\frac{17}{30}, \frac{1}{6}, \frac{23}{30})$	$\mathcal{X}_2 = \begin{pmatrix} 104 & 5000 & 49 \\ 26 & 120 & -7 \\ 676 & -16250 & 28 \end{pmatrix}$
34	$4fieldsx$	4	10/3	$\Lambda_1 = \mathrm{Diag}(\frac{31}{36}, \frac{7}{36}, \frac{1}{12}, \frac{19}{36})$	$\mathcal{X}_1 = \begin{pmatrix} 6 & 10880 & 91125 & 1250 \\ 4 & -194 & 729 & -25 \\ 1 & 17 & -2 & -8 \\ 13 & -884 & -11664 & 198 \end{pmatrix}$
			34/3	$\Lambda_1 = \mathrm{Diag}(\frac{19}{36}, \frac{31}{36}, -\frac{1}{4}, \frac{7}{36})$	$\mathcal{X}_1 = \begin{pmatrix} 54 & 50 & 18225 & 2500 \\ 703 & 78 & 215784 & -5624 \\ -1 & 1 & 3 & 1 \\ 65 & -13 & -729 & 121 \end{pmatrix}$

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Table B.2 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Λ_i	$\mathcal{X}_i$
35	$\overline{4fieldsx}$	4	14/3	$\Lambda_1 = \mathrm{Diag}(\frac{29}{36}, \frac{17}{36}, \frac{7}{12}, \frac{5}{36})$	$\mathcal{X}_1 = \begin{pmatrix} 14 & 1045 & 4860 & 17732 \\ 14 & -342 & 1215 & -806 \\ 3 & 57 & -5 & -465 \\ 7 & -19 & -243 & 93 \end{pmatrix}$
			38/3	$\Lambda_1 = \mathrm{Diag}(\frac{17}{36}, \frac{5}{36}, \frac{1}{4}, \frac{29}{36})$	$\mathcal{X}_1 = \begin{pmatrix} 108 & 1463 & 10206 & 22 \\ 28 & -132 & 729 & -2 \\ 10 & 38 & 0 & -1 \\ 1610 & -6118 & -67068 & 32 \end{pmatrix}$

Table B.3: The characters ch M^i of the irreducible VOA modules in the genus $\mathcal{G}(\mathcal{C}, c)$

No.	$\mathcal{C}$	n	c	Characters ch M^i
1	t_m	1	8	ch $M^1 = q^{2/3}\left(q^{-1} + 248 + 4124q + 34752q^2 + 213126q^3 + \cdots\right)$
			16	ch $M^1 = q^{1/3}\left(q^{-1} + 496 + 69752q + 2115008q^2 + 34670620q^3 + \cdots\right)$
2	qs_2	2	1	ch $M^1 = q^{23/24}\left(q^{-1} + 3 + 4q + 7q^2 + 13q^3 + \cdots\right)$ ch $M^2 = q^{5/24}\left(2 + 2q + 6q^2 + 8q^3 + \cdots\right)$
			9	ch $M^1 = q^{5/8}\left(q^{-1} + 251 + 4872q + 48123q^2 + 335627q^3 + \cdots\right)$ ch $M^2 = q^{-1/8}\left(2 + 498q + 8750q^2 + 79248q^3 + \cdots\right)$
3	$\overline{qs_2}$	2	7	ch $M^1 = q^{17/24}\left(q^{-1} + 133 + 1673q + 11914q^2 + 63252q^3 + \cdots\right)$ ch $M^2 = q^{11/24}\left(56 + 968q + 7504q^2 + 42616q^3 + \cdots\right)$
			15	ch $M^1 = q^{3/8}\left(q^{-1} + 381 + 38781q + 1010062q^2 + 14752518q^3 + \cdots\right)$ ch $M^2 = q^{1/8}\left(56 + 14856q + 478512q^2 + 7841752q^3 + \cdots\right)$
4	$Lee-Yang$	2	14/5	ch $M^1 = q^{53/60}\left(q^{-1} + 14 + 42q + 140q^2 + 350q^3 + \cdots\right)$ ch $M^2 = q^{17/60}\left(7 + 34q + 119q^2 + 322q^3 + \cdots\right)$
			54/5	ch $M^1 = q^{11/20}\left(q^{-1} + 262 + 7638q + 103044q^2 + 907932q^3 + \cdots\right)$ ch $M^2 = q^{-1/20}\left(7 + 1770q + 37419q^2 + 413314q^3 + \cdots\right)$

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Table B.3 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Characters ch M^i
5	$\overline{Lee-Yang}$	2	26/5	$\text{ch } M^1 = q^{47/60}\left(q^{-1} + 52 + 377q + 1976q^2 + 7852q^3 + ...\right)$ $\text{ch } M^2 = q^{23/60}\left(26 + 299q + 1702q^2 + 7475q^3 + \cdots\right)$
			66/5	$\text{ch } M^1 = q^{9/20}\left(q^{-1} + 300 + 17397q + 344672q^2 + 4072878q^3 + \cdots\right)$ $\text{ch } M^2 = q^{1/20}\left(26 + 6747q + 183078q^2 + 2566199q^3 + \cdots\right)$
6	qs_3	3	2	$\text{ch } M^1 = q^{11/12}\left(q^{-1} + 8 + 17q + 46q^2 + 98q^3 + \cdots\right)$ $\text{ch } M^2 = q^{1/4}\left(1 + 3q + 9q^2 + 19q^3 + \cdots\right)$ $\text{ch } M^3 = q^{1/4}\left(1 + 3q + 9q^2 + 19q^3 + \cdots\right)$
			10	$\text{ch } M^1 = q^{7/12}\left(q^{-1} + 256 + 6125q + 72006q^2 + 572756q^3 + \cdots\right)$ $\text{ch } M^2 = q^{-1/12}\left(1 + 251q + 4877q^2 + 49375q^3 + \cdots\right)$ $\text{ch } M^3 = q^{-1/12}\left(1 + 251q + 4877q^2 + 49375q^3 + \cdots\right)$
7	$\overline{qs_3}$	3	6	$\text{ch } M^1 = q^{3/4}\left(q^{-1} + 78 + 729q + 4382q^2 + 19917q^3 + \cdots\right)$ $\text{ch } M^2 = q^{5/12}\left(1 + 14q + 92q^2 + 456q^3 + \cdots\right)$ $\text{ch } M^3 = q^{5/12}\left(1 + 14q + 92q^2 + 456q^3 + \cdots\right)$
			14	$\text{ch } M^1 = q^{5/12}\left(q^{-1} + 326 + 24197q + 541598q^2 + 7036831q^3 + \cdots\right)$ $\text{ch } M^2 = q^{1/12}\left(1 + 262q + 7688q^2 + 115760q^3 + \cdots\right)$ $\text{ch } M^3 = q^{1/12}\left(1 + 262q + 7688q^2 + 115760q^3 + \cdots\right)$
8	$Ising1$	3	1/2	$\text{ch } M^1 = q^{47/48}\left(q^{-1} + q + q^2 + 2q^3 + \cdots\right)$ $\text{ch } M^2 = q^{23/48}\left(1 + q + q^2 + q^3 + \cdots\right)$ $\text{ch } M^3 = q^{1/24}\left(1 + q + q^2 + 2q^3 + \cdots\right)$
			17/2	$\text{ch } M^1 = q^{31/48}\left(q^{-1} + (136 + 112d) + (2669 + 1456d)q + (24361 + 10640d)q^2 + \cdots\right)$ $\text{ch } M^2 = q^{7/48}\left((17 - 16d) + (697 - 448d)q + (8517 - 4144d)q^2 + \cdots\right)$ $\text{ch } M^3 = q^{17/24}\left(dq^{-1} + (256 - 7d) + (4352 + 21d)q + (39168 - 42d)q^2 + \cdots\right)$ where d is a non negative integer
9	$\overline{Ising1}$	3	15/2	$\text{ch } M^1 = q^{11/16}\left(q^{-1} + 105 + 1590q + 12160q^2 + 69780q^3 + \cdots\right)$ $\text{ch } M^2 = q^{3/16}\left(15 + 470q + 4593q^2 + 30075q^3 + \cdots\right)$ $\text{ch } M^3 = q^{5/8}\left(128 + 1920q + 15360q^2 + 88960q^3 + \cdots\right)$
			31/2	$\text{ch } M^1 = q^{17/48}\left(q^{-1} + (248 + 7d) + (31124 + 42d)q + (871627 + 175d)q^2 + \cdots\right)$ $\text{ch } M^2 = q^{41/48}\left(dq^{-1} + (3875 + 21d) + (181753 + 84d)q + (3623869 + 322d)q^2 + \cdots\right)$ $\text{ch } M^3 = q^{7/24}\left((248 - 8d) + (34504 - 56d)q + (1022752 - 224d)q^2 + \cdots\right)$

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Table B.3 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Characters ch M^i
				where d is a non negative integer
10	$Ising2$	3	3/2	ch $M^1 = q^{15/16}\left(q^{-1} + 3 + 9q + 15q^2 + 30q^3 + \cdots\right)$ ch $M^2 = q^{7/16}\left(3 + 4q + 12q^2 + 21q^3 + \cdots\right)$ ch $M^3 = q^{1/8}\left(2 + 6q + 12q^2 + 26q^3 + \cdots\right)$
			19/2	ch $M^1 = q^{29/48}\left(q^{-1} + (171 + 40d) + (4237 + 320d)q + (46075 + 1648d)q^2 + \cdots\right)$ ch $M^2 = q^{5/48}\left((19 - 8d) + (988q - 120d)q + (14896 - 760d)q^2 + \cdots\right)$ ch $M^3 = q^{19/24}\left(dq^{-1} + (512 - 5d) + (9728q + 10d)q + (97280q^2 - 15d)q^2 + \cdots\right)$ where d is a non negative integer
11	$\overline{Ising2}$	3	13/2	ch $M^1 = q^{35/48}\left(q^{-1} + 78 + 884q + 5681q^2 + 28158q^3 + \cdots\right)$ ch $M^2 = q^{11/48}\left(13 + 299q + 2314q^2 + 13052q^3 + \cdots\right)$ ch $M^3 = q^{13/24}\left(64 + 832q + 5824q^2 + 29952q^3 + \cdots\right)$
			29/2	ch $M^1 = q^{19/48}\left(q^{-1} + (261 + 5d) + (24157 + 15d)q + (580609 + 56d)q^2 + \cdots\right)$ ch $M^2 = q^{43/48}\left(dq^{-1} + (3393 + 10d) + (129688 + 30d)q + (2270671 + 85d)q^2 + \cdots\right)$ ch $M^3 = q^{5/24}\left((116 - 4d) + (16964 - 20d)q + (476876 - 60d)q^2 + \cdots\right)$ where d is a non negative integer
12	$Ising3$	3	5/2	ch $M^1 = q^{43/48}\left(q^{-1} + 10 + 30q + 85q^2 + 205q^3 + \cdots\right)$ ch $M^2 = q^{19/48}\left(5 + 15q + 56q^2 + 130q^3 + \cdots\right)$ ch $M^3 = q^{5/24}\left(4 + 20q + 60q^2 + 160q^3 + \cdots\right)$
			21/2	ch $M^1 = q^{9/16}\left(q^{-1} + (210 + 12d) + (6426 + 52d)q + (82845 + 168d)q^2 + \cdots\right)$ ch $M^2 = q^{1/16}\left((21 - 4d) + (1351q - 24d)q + (24780 - 96d)q^2 + \cdots\right)$ ch $M^3 = q^{7/8}\left(dq^{-1} + (1024 - 3d) + (21504q + 3d)q + (236544q^2 - 4d)q^2 + \cdots\right)$ where d is a non negative integer
13	$\overline{Ising3}$	3	11/2	ch $M^1 = q^{37/48}\left(q^{-1} + 55 + 451q + 2453q^2 + 10329q^3 + \cdots\right)$ ch $M^2 = q^{13/48}\left(11 + 176q + 1078q^2 + 5181q^3 + \cdots\right)$ ch $M^3 = q^{11/24}\left(32 + 352q + 2112q^2 + 9504q^3 + \cdots\right)$
			27/2	ch $M^1 = q^{7/16}\left(q^{-1} + (270 + 3d) + (18171 + 4d)q + (375741 + 12d)q^2 + \cdots\right)$ ch $M^2 = q^{15/16}\left(dq^{-1} + (2871 + 3d) + (89991 + 9d)q + (1380456 + 15d)q^2 + \cdots\right)$ ch $M^3 = q^{1/8}\left((54 - 2d) + (8354 - 6d)q + (221508 - 12d)q^2 + \cdots\right)$

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Table B.3 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Characters ch M^i
				where d is a non negative integer
14	$Ising4$	3	7/2	ch $M^1 = q^{41/48}\left(q^{-1} + 21 + 84q + 322q^2 + 931q^3 + \cdots\right)$ ch $M^2 = q^{17/48}\left(7 + 42q + 175q^2 + 547q^3 + \cdots\right)$ ch $M^3 = q^{7/24}\left(8 + 56q + 224q^2 + 728q^3 + \cdots\right)$
			23/2	ch $M^1 = q^{25/48}\left(q^{-1} + (253 + 2d) + (9384 + 4d)q + (142462 + 6d)q^2 + \cdots\right)$ ch $M^2 = q^{1/48}\left((23 - 2d) + (1794 - 2d)q + (39491 - 4d)q^2 + \cdots\right)$ ch $M^3 = q^{23/24}\left(dq^{-1} + (2048 - d) + 47104q + (565248 - d)q^2 + \cdots\right)$ where d is a non negative integer
15	$\overline{Ising4}$	3	9/2	ch $M^1 = q^{13/16}\left(q^{-1} + 36 + 207q + 957q^2 + 3357q^3 + \cdots\right)$ ch $M^2 = q^{5/16}\left(9 + 93q + 459q^2 + 1827q^3 + \cdots\right)$ ch $M^3 = q^{3/8}\left(16 + 144q + 720q^2 + 2784q^3 + \cdots\right)$
			25/2	ch $M^1 = q^{23/48}\left(q^{-1} + (275 + d) + (13250 + d)q + (235500 + d)q^2 + \cdots\right)$ ch $M^2 = q^{47/48}\left(dq^{-1} + 2325 + (60630 + d)q + (811950 + d)q^2 + \cdots\right)$ ch $M^3 = q^{1/24}\left((25 - d) + (4121 - d)q + (102425 - d)q^2 + \cdots\right)$ where d is a non negative integer
16	$3fieldsx$	3	8/7	ch $M^1 = q^{-1/21}\left(q^{-1} + 14 + 66512q + 8878186q^2 + 405729320q^3 + \cdots\right)$ ch $M^2 = q^{17/21}\left(50922 + 8656740q + 441429616q^2 + 12203476160q^3 + \cdots\right)$ ch $M^3 = q^{5/21}\left(782 + 718267q + 64206178q^2 + 2419951472q^3 + \cdots\right)$
			64/7	ch $M^1 = q^{13/21}\left(q^{-1} + 136 + 2417q + 24520q^2 + 173412q^3 + \cdots\right)$ ch $M^2 = q^{10/21}\left(117 + 2952q + 32220q^2 + 239680q^3 + \cdots\right)$ ch $M^3 = q^{-2/21}\left(3 + 632q + 10787q^2 + 98104q^3 + \cdots\right)$
17	$\overline{3fieldsx}$	3	48/7	ch $M^1 = q^{5/7}\left(q^{-1} + 78 + 784q + 5271q^2 + 26558q^3 + \cdots\right)$ ch $M^2 = q^{-1/7}\left(1 + 133q + 1618q^2 + 11024q^3 + \cdots\right)$ ch $M^3 = q^{3/7}\left(55 + 890q + 6720q^2 + 37344q^3 + \cdots\right)$
			104/7	ch $M^1 = q^{8/21}\left(q^{-1} + (188 + 138d) + (17260 + 6992d)q + (442300 + 113827d)q^2 + \cdots\right)$ ch $M^2 = q^{11/21}\left(dq^{-1} + (725 - 344d) + (52316 - 13590d)q + (1197468 - 201936d)q^2 + \cdots\right)$ ch $M^3 = q^{2/21}\left((44 + 11d) + (13002 + 1528d)q + (424040 + 30220d)q^2 + \cdots\right)$ where d is a non negative integer

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Table B.3 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Characters ch M^i
18	qs_4	4	1	$\text{ch } M^1 = q^{23/24}\left(q^{-1} + 1 + 4q + 5q^2 + 9q^3 + \cdots\right)$ $\text{ch } M^2 = q^{11/24}\left(2 + 2q + 4q^2 + 6q^3 + \cdots\right)$ $\text{ch } M^3 = q^{1/12}\left(1 + 2q + 3q^2 + 6q^3 + \cdots\right)$ $\text{ch } M^4 = q^{1/12}\left(1 + 2q + 3q^2 + 6q^3 + \cdots\right)$
			9	$\text{ch } M^1 = q^{5/8}\left(q^{-1} + (153 + 96d) + (3384 + 992d)q + (33729 + 6144d)q^2 + \cdots\right)$ $\text{ch } M^2 = q^{1/8}\left((18 - 16d) + (834 - 336d)q + (11340 - 2592d)q^2 + \cdots\right)$ $\text{ch } M^3 = q^{3/4}\left(dq^{-1} + (256 - 6d) + (4608 + 15d)q + (43776 - 26d)q^2 + \cdots\right)$ $\text{ch } M^4 = q^{3/4}\left(dq^{-1} + (256 - 6d) + (4608 + 15d)q + (43776 - 26d)q^2 + \cdots\right)$ where d is a non negative integer
19	$\overline{qs_4}$	4	7	$\text{ch } M^1 = q^{17/24}\left(q^{-1} + 91 + 1197q + 8386q^2 + 44800q^3 + \cdots\right)$ $\text{ch } M^2 = q^{5/24}\left(14 + 378q + 3290q^2 + 20008q^3 + \cdots\right)$ $\text{ch } M^3 = q^{7/12}\left(64 + 896q + 6720q^2 + 36736q^3 + \cdots\right)$ $\text{ch } M^4 = q^{7/12}\left(64 + 896q + 6720q^2 + 36736q^3 + \cdots\right)$
			15	$\text{ch } M^1 = q^{3/8}\left(q^{-1} + (255 + 6d) + (27525 + 26d)q + (713850 + 102d)q^2 + \cdots\right)$ $\text{ch } M^2 = q^{7/8}\left(dq^{-1} + (3640 + 15d) + (154056 + 51d)q + (2878920 + 172d)q^2 + \cdots\right)$ $\text{ch } M^3 = q^{1/4}\left((120 - 4d) + (17104 - 24d)q + (494040 - 84d)q^2 + \cdots\right)$ $\text{ch } M^4 = q^{1/4}\left((120 - 4d) + (17104 - 24d)q + (494040 - 84d)q^2 + \cdots\right)$ where d is a non negative integer
20	qn_4	4	5	$\text{ch } M^1 = q^{19/24}\left(q^{-1} + 45 + 310q + 1555q^2 + 5990q^3 + \cdots\right)$ $\text{ch } M^2 = q^{7/24}\left(10 + 130q + 712q^2 + 3130q^3 + \cdots\right)$ $\text{ch } M^3 = q^{5/12}\left(16 + 160q + 880q^2 + 3680q^3 + \cdots\right)$ $\text{ch } M^4 = q^{5/12}\left(16 + 160q + 880q^2 + 3680q^3 + \cdots\right)$
			13	$\text{ch } M^1 = q^{11/24}\left(q^{-1} + (273 + 2d) + (15574 + 2d)q + (298727 + 4d)q^2 + \cdots\right)$ $\text{ch } M^2 = q^{23/24}\left(dq^{-1} + (2600 + d) + (74152 + 4d)q + (1063296 + 5d)q^2 + \cdots\right)$ $\text{ch } M^3 = q^{1/12}\left((26 - d) + (4148 - 2d)q + (106574 - 3d)q^2 + \cdots\right)$ $\text{ch } M^4 = q^{1/12}\left((26 - d) + (4148 - 2d)q + (106574 - 3d)q^2 + \cdots\right)$ where d is a non negative integer

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Table B.3 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Characters ch M^i
21	$\overline{qn_4}$	4	3	ch $M^1 = q^{7/8}\left(q^{-1} + 15 + 51q + 172q^2 + 453q^3 + \cdots\right)$ ch $M^1 = q^{3/8}\left(6 + 26q + 102q^2 + 276q^3 + \cdots\right)$ ch $M^2 = q^{1/4}\left(4 + 24q + 84q^2 + 248q^3 + \cdots\right)$ ch $M^3 = q^{1/4}\left(4 + 24q + 84q^2 + 248q^3 + \cdots\right)$
			11	ch $M^1 = q^{13/24}\left(q^{-1} + (231 + 8d) + (7799 + 24d)q + (109208 + 56d)q^2 + \cdots\right)$ ch $M^2 = q^{1/24}\left((22 - 4d) + (1562 - 12d)q + (31438 - 36d)q^2 + \cdots\right)$ ch $M^3 = q^{11/12}\left(dq^{-1} + (1024 - 2d) + (22528 + d)q + (259072 - 2d)q^2 + \cdots\right)$ ch $M^4 = q^{11/12}\left(dq^{-1} + (1024 - 2d) + (22528 + d)q + (259072 - 2d)q^2 + \cdots\right)$ where d is a non negative integer
22	qu_2	4	8	ch $M^1 = q^{2/3}\left(q^{-1} + 120 + 2076q + 17344q^2 + 106630q^3 + \cdots\right)$ ch $M^2 = q^{2/3}\left(128 + 2048q + 17408q^2 + 106496q^3 + \cdots\right)$ ch $M^3 = q^{2/3}\left(128 + 2048q + 17408q^2 + 106496q^3 + \cdots\right)$ ch $M^4 = q^{1/6}\left(16 + 576q + 6304q^2 + 44416q^3 + \cdots\right)$
			16	ch $M^1 = q^{1/3}\left(q^{-1} + (240 + 32d) + (34936 + 256d)q + (1057216 + 1152d)q^2 + \cdots\right)$ ch $M^2 = q^{1/3}\left((256 + 32d) + (34816 + 256d)q + (1057792 + 1152d)q^2 + \cdots\right)$ ch $M^3 = q^{1/3}\left((256 + 32d) + (34816 + 256d)q + (1057792 + 1152d)q^2 + \cdots\right)$ ch $M^4 = q^{5/6}\left(dq^{-1} + (1024 + 28d) + (53248 + 134d)q + (1132544 + 568d)q^2 + \cdots\right)$ where d is a non negative integer
23	qv_2	4	4	ch $M^1 = q^{5/6}\left(q^{-1} + 28 + 134q + 568q^2 + 1809q^3 + \cdots\right)$ ch $M^2 = q^{1/3}\left(8 + 64q + 288q^2 + 1024q^3 + \cdots\right)$ ch $M^3 = q^{1/3}\left(8 + 64q + 288q^2 + 1024q^3 + \cdots\right)$ ch $M^4 = q^{1/3}\left(8 + 64q + 288q^2 + 1024q^3 + \cdots\right)$
			12	ch $M^1 = q^{1/2}\left(q^{-1} + 276 + 11202q + 184024q^2 + 1881471q^3 + \cdots\right)$ ch $M^2 = q\left(8 + 2048q + 49152q^2 + 614400q^3 + \cdots\right)$ ch $M^3 = q\left(8 + 2048q + 49152q^2 + 614400q^3 + \cdots\right)$ ch $M^4 = q\left(8 + 2048q + 49152q^2 + 614400q^3 + \cdots\right)$

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Table B.3 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Characters ch M^i
24	$qs_2 \otimes qs_2$	4	2	ch $M^1 = q^{11/12}\left(q^{-1} + 6 + 17q + 38q^2 + 84q^3 + \cdots\right)$ ch $M^2 = q^{1/6}\left(2 + 8q + 20q^2 + 48q^3 + \cdots\right)$ ch $M^3 = q^{1/6}\left(2 + 8q + 20q^2 + 48q^3 + \cdots\right)$ ch $M^4 = q^{5/12}\left(4 + 8q + 28q^2 + 56q^3 + \cdots\right)$
			10	ch $M^1 = q^{7/12}\left(q^{-1} + (190 + 32d) + (5245 + 192d)q + (62150 + 800d)q^2 + \cdots\right)$ ch $M^2 = q^{5/6}\left(dq^{-1} + (512 - 4d) + (10240 + 6d)q + (107520 - 8d)q^2 + \cdots\right)$ ch $M^3 = q^{5/6}\left(dq^{-1} + (512 - 4d) + (10240 + 6d)q + (107520 - 8d)q^2 + \cdots\right)$ ch $M^4 = q^{1/12}\left((20 - 8d) + (1160 - 80d)q + (19324 - 408d)q^2 + \cdots\right)$ where d is a non negative integer
25	$\overline{qs_2} \otimes \overline{qs_2}$	4	6	ch $M^1 = q^{3/4}\left(q^{-1} + 66 + 639q + 3774q^2 + 17283q^3 + \cdots\right)$ ch $M^2 = q^{1/2}\left(32 + 384q + 2496q^2 + 12032q^3 + \cdots\right)$ ch $M^3 = q^{1/2}\left(32 + 384q + 2496q^2 + 12032q^3 + \cdots\right)$ ch $M^4 = q^{1/4}\left(12 + 232q + 1596q^2 + 8328q^3 + \cdots\right)$
			14	ch $M^1 = q^{5/12}\left(q^{-1} + 266 + 21035q + 468846q^2 + 6094557q^3 + \cdots\right)$ ch $M^2 = q^{1/6}\left(56 + 8416q + 229936q^2 + 3327296q^3 + \cdots\right)$ ch $M^3 = q^{1/6}\left(56 + 8416q + 229936q^2 + 3327296q^3 + \cdots\right)$ ch $M^4 = q^{11/12}\left(3136 + 108416q + 1777472q^2 + 19300736q^3 + \cdots\right)$
26	$qs_2 \otimes \overline{qs_2}$	4	8	ch $M^1 = q^{2/3}\left(q^{-1} + 136 + 2076q + 17472q^2 + 106630q^3 + \cdots\right)$ ch $M^2 = q^{-1/12}\left(2 + 268q + 3618q^2 + 27980q^3 + \cdots\right)$ ch $M^3 = q^{5/12}\left(56 + 1136q + 10632q^2 + 69392q^3 + \cdots\right)$ ch $M^4 = q^{2/3}\left(112 + 2048q + 17280q^2 + 106496q^3 + \cdots\right)$
			16	ch $M^1 = q^{1/3}\left(q^{-1} + (256 + 64d) + (34808 + 2560d)q + (1057792 + 35072d)q^2 + \cdots\right)$ ch $M^2 = q^{1/12}\left((32 + 12d) + (12608 + 1208d)q + (484960 + 19172d)q^2 + \cdots\right)$ ch $M^3 = q^{7/12}\left(dq^{-1} + (1152 - 194d) + (88832 - 5251d)q + (2224256 - 62138d)q^2 + \cdots\right)$ ch $M^4 = q^{1/3}\left((240 + 64d) + (34944 + 2560d)q + (1057216 + 35072d)q^2 + \cdots\right)$ where d is a non negative integer

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Table B.3 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Characters ch M^i
27	$qs_2 \otimes LY$	4	19/5	ch $M^1 = q^{101/120}\left(q^{-1} + 17 + 88q + 329q^2 + 1049q^3 + \cdots\right)$ ch $M^2 = q^{29/120}\left(7 + 55q + 249q^2 + 864q^3 + \cdots\right)$ ch $M^3 = q^{11/120}\left(2 + 30q + 118q^2 + 456q^3 + \cdots\right)$ ch $M^4 = q^{59/120}\left(14 + 82q + 348q^2 + 1142q^3 + \cdots\right)$
			59/5	ch $M^1 = q^{61/120}\left(q^{-1} + (193 + 36d) + (7872 + 278d)q + (123649 + 1682d)q^2 + \cdots\right)$ ch $M^2 = q^{-11/120}\left((3 + 2d) + (1603 + 94d)q + (41017 + 870d)q^2 + \cdots\right)$ ch $M^3 = q^{91/120}\left(dq^{-1} + (592 - 33d) + 16536q - 365dq + 227464q^2 - 2260dq^2 + \cdots\right)$ ch $M^4 = q^{19/120}\left((40 - 13d) + (3976 - 211d)q + (81296 - 1438d)q^2 + \cdots\right)$ where d is a non negative integer
28	$\overline{qs_2} \otimes LY$	4	9/5	ch $M^1 = q^{37/40}\left(q^{-1} + 3 + 9q + 22q^2 + 42q^3 + \cdots\right)$ ch $M^2 = q^{12/40}\left(3 + 9q + 20q^2 + 45q^3 + \cdots\right)$ ch $M^3 = q^{27/40}\left(4 + 6q + 18q^2 + 34q^3 + \cdots\right)$ ch $M^4 = q^{3/40}\left(2 + 6q + 18q^2 + 36q^3 + \cdots\right)$
			49/5	ch $M^1 = q^{71/120}\left(q^{-1} + 147 + 3577q + 41062q^2 + 319284q^3 + \cdots\right)$ ch $M^2 = q^{-1/120}\left(7 + 965q + 16352q^2 + 156429q^3 + \cdots\right)$ ch $M^3 = q^{41/120}\left(56 + 1752q + 23408q^2 + 196168q^3 + \cdots\right)$ ch $M^4 = q^{89/120}\left(392 + 8680q + 92104q^2 + 686672q^3 + \cdots\right)$
29	$qs_2 \otimes \overline{LY}$	4	31/5	ch $M^1 = q^{89/120}\left(q^{-1} + 55 + 537q + 3322q^2 + 15665q^3 + \cdots\right)$ ch $M^2 = q^{41/120}\left(26 + 377q + 2703q^2 + 13959q^3 + \cdots\right)$ ch $M^3 = q^{-1/120}\left(2 + 106q + 864q^2 + 5026q^3 + \cdots\right)$ ch $M^4 = q^{71/120}\left(52 + 650q + 4158q^2 + 20356q^3 + \cdots\right)$
			71/5	ch $M^1 = q^{49/120}\left(q^{-1} + (211 + 46d) + (16529 + 886d)q + (380042 + 9014d)q^2 + \cdots\right)$ ch $M^2 = q^{1/120}\left((14 + 6d) + (5837 + 494d)q + (191143 + 6140d)q^2 + \cdots\right)$ ch $M^3 = q^{79/120}\left(dq^{-1} + (792 - 95d) + (38792 - 1696d)q + (755648 - 14851d)q^2 + \cdots\right)$ ch $M^4 = q^{31/120}\left((120 - 34d) + (15528 - 991d)q + (400984 - 10589d)q^2 + \cdots\right)$ where d is a non negative integer

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Table B.3 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Characters ch M^i
30	$\overline{qs_2} \otimes \overline{LY}$	4	21	ch $M^1 = q^{33/40}\left(q^{-1} + 21 + 126q + 511q^2 + 1743q^3 + \cdots\right)$ ch $M^2 = q^{17/40}\left(14 + 105q + 483q^2 + 1764q^3 + \cdots\right)$ ch $M^3 = q^{23/40}\left(14 + 78q + 378q^2 + 1288q^3 + \cdots\right)$ ch $M^4 = q^{7/40}\left(6 + 70q + 336q^2 + 1302q^3 + \cdots\right)$
			61/5	ch $M^1 = q^{59/120}\left(q^{-1} + (185 + 14d) + (8966 + 82d)q + (151027 + 348d)q^2 + \cdots\right)$ ch $M^2 = q^{11/120}\left((26 - 2d) + (3757 - 30d)q + (84967 - 118d)q^2 + \cdots\right)$ ch $M^3 = q^{29/120}\left((56 - 7d) + (3880 - 55d)q + (78952 - 249d)q^2 + \cdots\right)$ ch $M^4 = q^{101/120}\left(dq^{-1} + (1456 + 17d) + (41912 + 88d)q + (579848 + 329d)q^2 + \cdots\right)$ where d is a non negative integer
31	$LY \otimes LY$	4	28/5	ch $M^1 = q^{23/30}\left(q^{-1} + 28 + 280q + 1456q^2 + 6384q^3 + \cdots\right)$ ch $M^2 = q^{1/6}\left(7 + 132q + 889q^2 + 4396q^3 + \cdots\right)$ ch $M^3 = q^{1/6}\left(7 + 132q + 889q^2 + 4396q^3 + \cdots\right)$ ch $M^4 = q^{17/30}\left(49 + 476q + 2822q^2 + 12600q^3 + \cdots\right)$
			68/5	ch $M^1 = q^{13/30}\left(q^{-1} + (136 + 20d) + (10438 + 130d)q + (216920 + 600d)q^2 + \cdots\right)$ ch $M^2 = q^{5/6}\left(dq^{-1} + (1700 + 24d) + (61625 + 124d)q + (1009000 + 500d)q^2 + \cdots\right)$ ch $M^3 = q^{5/6}\left(dq^{-1} + (1700 + 24d) + (61625 + 124d)q + (1009000 + 500d)q^2 + \cdots\right)$ ch $M^4 = q^{7/30}\left((119 - 10d) + (13328 - 100d)q + (326026 - 440d)q^2 + \cdots\right)$ where d is a non negative integer
32	$LY \otimes \overline{LY}$	4	8	ch $M^1 = q^{2/3}\left(q^{-1} + 66 + 1147q + 9578q^2 + 58980q^3 + \cdots\right)$ ch $M^2 = q^{4/15}\left(52 + 1326q + 13960q^2 + 95002q^3 + \cdots\right)$ ch $M^3 = q^{1/15}\left(14 + 796q + 9052q^2 + 66320q^3 + \cdots\right)$ ch $M^4 = q^{2/3}\left(182 + 2977q + 25174q^2 + 154146q^3 + \cdots\right)$
			16	ch $M^1 = q^{1/3}\left(q^{-1} + (139 + 25d) + (19364 + 325d)q + (584345 + 2375d)q^2 + \cdots\right)$ ch $M^2 = q^{-1/15}\left((5 + 3d) + (5795 + 188d)q + (266350 + 1754d)q^2 + \cdots\right)$ ch $M^3 = q^{11/15}\left(dq^{-1} + (2295 - 23d) + (135150 - 436d)q + (3059880 - 2808d)q^2 + \cdots\right)$ ch $M^4 = q^{1/3}\left((357 + 25d) + (50388 + 325d)q + (1530663 + 2375d)q^2 + \cdots\right)$ where d is a non negative integer

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Table B.3 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	Characters ch M^i
33	$\overline{LY} \otimes \overline{LY}$	4	12/5	$\text{ch } M^1 = q^{9/10}\left(q^{-1} + 3 + 18q + 38q^2 + 99q^3 + \cdots\right)$ $\text{ch } M^2 = q^{1/2}\left(5 + 15q + 45q^2 + 110q^3 + \cdots\right)$ $\text{ch } M^3 = q^{1/2}\left(5 + 15q + 45q^2 + 110q^3 + \cdots\right)$ $\text{ch } M^4 = q^{1/10}\left(3 + 16q + 48q^2 + 129q^3 + \cdots\right)$
			52/5	$\text{ch } M^1 = q^{17/30}\left(q^{-1} + (104 + 49d) + (3458 + 476d)q + (43160 + 2822d)q^2 + \cdots\right)$ $\text{ch } M^2 = q^{1/6}\left((26 - 7d) + (1651 - 132d)q + (27052 - 889d)q^2 + \cdots\right)$ $\text{ch } M^3 = q^{1/6}\left((26 - 7d) + (1651 - 132d)q + (27052 - 889d)q^2 + \cdots\right)$ $\text{ch } M^4 = q^{23/30}\left(dq^{-1} + (676 + 28d) + (15548 + 280d)q + (177905 + 1456d)q^2 + \cdots\right)$ where d is a non negative integer
34	$4fieldsx$	4	10/3	$\text{ch } M^1 = q^{31/36}\left(q^{-1} + 6 + 38q + 112q^2 + 347q^3 + \cdots\right)$ $\text{ch } M^2 = q^{7/36}\left(4 + 23q + 102q^2 + 319q^3 + \cdots\right)$ $\text{ch } M^3 = q^{1/12}\left(3 + 30q + 114q^2 + 384q^3 + \cdots\right)$ $\text{ch } M^4 = q^{19/36}\left(13 + 62q + 230q^2 + 692q^3 + \cdots\right)$
			34/3	$\text{ch } M^1 = q^{19/36}\left(q^{-1} + (54 + 50d) + (3630 + 505d)q + (56308 + 3181d)q^2 + \cdots\right)$ $\text{ch } M^2 = q^{31/36}\left(dq^{-1} + (703 + 78d) + (19018 + 821d)q + (240019 + 4864d)q^2 + \cdots\right)$ $\text{ch } M^3 = q^{-1/4}\left((-1 + d) + (714 + 15d)q + (19602 + 81d)q^2 + \cdots\right)$ $\text{ch } M^4 = q^{7/36}\left((65 - 13d) + (4278 - 248d)q + (76142 - 1731d)q^2 + \cdots\right)$ where d is a non negative integer
35	$\overline{4fieldsx}$	4	14/3	$\text{ch } M^1 = q^{29/36}\left(q^{-1} + 14 + 119q + 497q^2 + 1890q^3 + \cdots\right)$ $\text{ch } M^2 = q^{17/36}\left(14 + 119q + 588q^2 + 2331q^3 + \cdots\right)$ $\text{ch } M^3 = q^{7/12}\left(3 + 21q + 105q^2 + 399q^3 + \cdots\right)$ $\text{ch } M^4 = q^{5/36}\left(7 + 98q + 547q^2 + 2310q^3 + \cdots\right)$
			38/3	$\text{ch } M^1 = q^{17/36}\left(q^{-1} + (108 + 22d) + (6469 + 178d)q + (116092 + 915d)q^2 + \cdots\right)$ $\text{ch } M^2 = q^{5/36}\left((28 - 2d) + (3850 - 37d)q + (89110 - 182d)q^2 + \cdots\right)$ $\text{ch } M^3 = q^{1/4}\left((10 - d) + (849 - 12d)q + (18126 - 63d)q^2 + \cdots\right)$ $\text{ch } M^4 = q^{29/36}\left(dq^{-1} + (1610 + 32d) + (52256 + 209d)q + (778690 + 956d)q^2 + \cdots\right)$ where d is a non negative integer

Table B.4: The possible subVOAs $\tilde{V}_1$ in the genus $\mathcal{G}(\mathcal{C}, c)$

No.	$\mathcal{C}$	n	c	subVOAs $\tilde{V}_1$
1	t_m	1	8	$E_{8,1}$
			16	$E_{8,1} \otimes E_{8,1}$, $D_{16,1}$
2	qs_2	2	1	$A_{1,1}$
			9	$A_{1,1} \otimes E_{8,1}$
3	$\overline{qs_2}$	2	7	$E_{7,1}$
			15	$E_{7,1} \otimes E_{8,1}$, $A_{1,1} \otimes D_{14,1}$
4	$Lee-Yang$	2	14/5	$G_{2,1}$
			54/5	$G_{2,1} \otimes E_{8,1}$
5	$\overline{Lee-Yang}$	2	26/5	$F_{4,1}$
			66/5	$F_{4,1} \otimes E_{8,1}$, $B_{12,1}(7/10)$
6	qs_3	3	2	$A_{2,1}$
			10	$A_{2,1} \otimes E_{8,1}$
7	$\overline{qs_3}$	3	6	$E_{6,1}$
			14	$E_{6,1} \otimes E_{8,1}$
8	$Ising1$	3	1/2	(1/2)
			17/2	$B_{8,1}$, $A_{1,1} \otimes E_{7,1}(1/2)$, $A_{1,2} \otimes E_{7,1}$, $E_{8,1}(1/2)$
9	$\overline{Ising1}$	3	15/2	$B_{7,1}$
			31/2	$D_{12,1}(7/2)$, $D_{13,1}(5/2)$, and 33 more
10	$Ising2$	3	3/2	$A_{1,2}$, $A_{1,1}(1/2)$
			19/2	$B_{9,1}$, $A_{1,2} \otimes E_{8,1}$, $A_{1,1} \otimes E_{8,1}(1/2)$
11	$\overline{Ising2}$	3	13/2	$B_{6,1}$, $E_{6,1}(1/2)$
			29/2	$D_{12,1}(5/2)$, $B_{13,1}(1)$ and 8 more
12	$Ising3$	3	5/2	$B_{2,1}$
			21/2	$B_{10,1}$, $B_{2,1} \otimes E_{8,1}$
13	$\overline{Ising3}$	3	11/2	$B_{5,1}$
			27/2	$D_{12,1}(3/2)$, $B_{12,1}(1)$, $A_{1,2} \otimes D_{12,1}$, $B_{5,1} \otimes E_{8,1}$, $A_{1,1} \otimes D_{12,1}(1/2)$, $B_{13,1}$, $A_{1,1} \otimes B_{12,1}$

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Table B.4 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	subVOAs $\tilde{V}_1$
14	$Ising4$	3	7/2	$B_{3,1}$
			23/2	$B_{11,1}$, $B_{3,1} \otimes E_{8,1}$
15	$\overline{Ising4}$	3	9/2	$B_{4,1}$
			25/2	$D_{12,1}(1/2)$, $B_{4,1} \otimes E_{8,1}$, $B_{12,1}$
16	$3fieldsx$	3	8/7	None
			64/7	$B_{8,1}(9/14)$, $A_{1,2} \otimes E_{7,1}(9/14)$, $A_{1,3} \otimes E_{7,1}(12/35)$, $A_{1,4} \otimes E_{7,1}(1/7)$, $A_{1,5} \otimes E_{7,1}$
17	$\overline{3fieldsx}$	3	48/7	$E_{6,1}(6/7)$, $B_{6,1}(5/14)$
			104/7	$B_{5,1} \otimes E_{7,1}(33/14)$, $B_{8,1} \otimes F_{4,1}(81/70)$ and 32 more
18	qs_4	4	1	$\tilde{V}_{H_1}$
			9	$D_{9,1}$, $\tilde{V}_{H_1} \otimes E_{8,1}$
19	$\overline{qs_4}$	4	7	$D_{7,1}$
			15	$B_{13,1}(3/2)$, $A_{1,3} \otimes D_{12,1}(6/5)$, and 37 more
20	qn_4	4	5	$D_{5,1}$
			13	$H_1 \otimes D_{12,1}$, $A_{1,1} \otimes D_{12,1}$, $D_{5,1} \otimes E_{8,1}$, $D_{13,1}$
21	$\overline{qn_4}$	4	3	$A_{3,1}$
			11	$D_{11,1}$, $A_{3,1} \otimes E_{8,1}$
22	qu_2	4	8	$D_{8,1}$
			16	$A_{1,1} \otimes B_{12,1} \otimes \tilde{V}_{H_1}(3/2)$, $A_{1,1}^{\otimes 2} \otimes A_{1,2} \otimes D_{11,1}(3/2)$, and 47 more
23	qv_2	4	4	$D_{4,1}$
			12	$D_{4,1} \otimes E_{8,1}$, $D_{12,1}$
24	$qs_2 \otimes qs_2$	4	2	$A_{1,1}^{\otimes 2}$
			10	$D_{10,1}$, $A_{1,1}^{\otimes 2} \otimes E_{8,1}$
25	$\overline{qs_2} \otimes \overline{qs_2}$	4	6	$D_{6,1}$
			14	$E_{7,1}^{\otimes 2}$, $\tilde{V}_{H_1}^{\otimes 2} \otimes D_{12,1}$, $A_{1,1}^{\otimes 2} \otimes D_{12,1}$, $D_{6,1} \otimes E_{8,1}$, $D_{14,1}$, $\tilde{V}_{H_1} \otimes D_{13,1}$
26	$qs_2 \otimes \overline{qs_2}$	4	8	$A_{1,1} \otimes E_{7,1}$
			16	$A_{1,1} \otimes B_{11,1}(7/2)$, $A_{1,4} \otimes B_{11,1}(5/2)$, and 17 more
27	$qs_2 \otimes LY$	4	19/5	$A_{1,1} \otimes G_{2,1}$

Continued on next page

Table B.4 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	subVOAs $\tilde{V}_1$
			59/5	$A_{1,1} \otimes D_{10,1}(4/5)$, $A_{1,2} \otimes D_{10,1}(3/10)$, $A_{1,3} \otimes D_{10,1}$, $A_{1,1} \otimes G_{2,1} \otimes E_{8,1}$
28	$\overline{qs_2} \otimes LY$	4	9/5	$A_{1,3}$, $A_{1,1}(4/5)$
			49/5	$E_{7,1} \otimes G_{2,1}$, $A_{1,3} \otimes E_{8,1}$, $A_{1,1} \otimes E_{8,1}(4/5)$
29	$qs_2 \otimes \overline{LY}$	4	31/5	$A_{1,1} \otimes F_{4,1}$, $B_{5,1}(7/10)$
			71/5	$B_{3,1} \otimes D_{10,1}(7/10)$, $B_{6,1} \otimes E_{7,1}(7/10)$, $C_{3,1} \otimes D_{10,1}$, $A_{1,1} \otimes F_{4,1} \otimes E_{8,1}$, $A_{1,1} \otimes B_{11,1} \otimes \tilde{V}_{H_1}(7/10)$, $A_{1,1} \otimes B_{12,1}(7/10)$, $A_{1,2} \otimes B_{12,1}(1/5)$, $B_{5,1} \otimes E_{8,1}(7/10)$
30	$\overline{qs_2} \otimes \overline{LY}$	4	21/5	$C_{3,1}$, $B_{3,1}(7/10)$
			61/5	$F_{4,1} \otimes E_{7,1}$, $C_{3,1} \otimes E_{8,1}$, $A_{1,1} \otimes B_{10,1}(7/10)$, $A_{1,2} \otimes B_{10,1}(1/5)$, $B_{3,1} \otimes E_{8,1}(7/10)$
31	$LY \otimes LY$	4	28/5	$G_{2,1}^{\otimes 2}$
			68/5	$B_{8,1}(51/10)$, $D_{12,1}(8/5)$, and 96 more
32	$LY \otimes \overline{LY}$	4	8	$G_{2,1} \otimes F_{4,1}$, $A_{1,1} \otimes A_{7,1}$
			16	$A_{1,1}, \otimes B_{8,1}(13/2)$, $A_{1,1} \otimes C_{8,1}(7/5)$, and 953 more
33	$\overline{LY} \otimes \overline{LY}$	4	12/5	$A_{1,2}(9/10)$, $A_{1,3}(3/5)$, $A_{1,4}(2/5)$, $A_{1,5}(9/35)$, $A_{1,6}(3/20)$, $A_{1,7}(1/15)$, $A_{1,8}$
			52/5	$F_{4,1}^{\otimes 2}$, $A_{1,2} \otimes E_{8,1}(9/10)$, $A_{1,3} \otimes E_{8,1}(3/5)$, $A_{1,4} \otimes E_{8,1}(2/5)$, $A_{1,5} \otimes E_{8,1}(9/35)$, $A_{1,6} \otimes E_{8,1}(3/20)$, $A_{1,7} \otimes E_{8,1}(1/15)$, $A_{1,8} \otimes E_{8,1}$
34	$4fieldsx$	4	10/3	$A_{1,1} \otimes A_{1,7}$, $A_{1,1} \otimes A_{1,2}(5/6)$, $A_{1,1} \otimes A_{1,3}(8/15)$, $A_{1,1} \otimes A_{1,4}(1/3)$, $A_{1,1} \otimes A_{1,5}(4/21)$, $A_{1,1} \otimes A_{1,6}(1/12)$, $A_{1,2} \otimes A_{1,2}(1/3)$, $A_{1,2} \otimes A_{1,3}(1/30)$
			34/3	$A_{1,1}^{\otimes 2} \otimes A_{6,1}(10/3)$, $A_{1,1} \otimes A_{1,2} \otimes A_{6,1}(17/6)$, and 384 more
35	$\overline{4fieldsx}$	4	14/3	$G_{2,2}$, $A_{1,1}^{\otimes 2} \otimes A_{1,2}(2/3)$, $A_{1,1} \otimes A_{1,2} \otimes A_{2,1}(1/6)$ $A_{1,1} \otimes B_{2,1} \otimes \tilde{V}_{H_1}(1/6)$
			38/3	$A_{1,1} \otimes B_{9,1}(13/6)$, $A_{1,1} \otimes B_{7,1}(25/6)$, and 84 more

Table B.5: The VOAs in the genus $\mathcal{G}(\mathcal{C}, c)$

No.	$\mathcal{C}$	n	c	VOAs	Method
1	t_m	1	8	$E_{8,1}$	1
			16	$E_{8,1} \otimes E_{8,1},\ (D_{16,1})^+$	2
2	qs_2	2	1	$A_{1,1}$	1
			9	$A_{1,1} \otimes E_{8,1}$	1
3	$\overline{qs_2}$	2	7	$E_{7,1}$	1
			15	$E_{7,1} \otimes E_{8,1},\ (A_{1,1} \otimes D_{14,1})^+$	3
4	$Lee-Yang$	2	14/5	$G_{2,1}$	1
			54/5	$G_{2,1} \otimes E_{8,1}$	1
5	$\overline{Lee-Yang}$	2	26/5	$F_{4,1}$	1
6	qs_3	3	2	$A_{2,1}$	1
			10	$A_{2,1} \otimes E_{8,1}$	1
7	$\overline{qs_3}$	3	6	$E_{6,1}$	1
			14	$E_{6,1} \otimes E_{8,1}$	1
8	$Ising1$	3	1/2	$L_{1/2}(0)$	2
			17/2	$B_{8,1},\ E_{8,1} \otimes L_{1/2}(0)$	2
9	$\overline{Ising1}$	3	15/2	$B_{7,1}$	1
10	$Ising2$	3	3/2	$A_{1,2}$	2
			19/2	$B_{9,1},\ A_{1,2} \otimes E_{8,1}$	2
11	$\overline{Ising2}$	3	13/2	$B_{6,1}$	2
12	$Ising3$	3	5/2	$B_{2,1}$	1
			21/2	$B_{10,1},\ B_{2,1} \otimes E_{8,1}$	1
13	$\overline{Ising3}$	3	11/2	$B_{5,1}$	1
14	$Ising4$	3	7/2	$B_{3,1}$	1

Continued on next page

Table B.5 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	VOAs	Method
			23/2	$B_{11,1},\ B_{3,1} \otimes E_{8,1}$	1
15	$\overline{Ising4}$	3	9/2	$B_{4,1}$	1
			25/2	$(D_{12,1} \otimes L_{1/2}(0))^+,\ B_{4,1} \otimes E_{8,1},\ B_{12,1}$	2
16	$3fieldsx$	3	8/7	None	4
			64/7	$A_{1,5} \otimes E_{7,1}$	2
17	$\overline{3fieldsx}$	3	48/7	cannot determine	
18	qs_4	4	1	$D_{1,1}$	3^*
			9	$D_{9,1},\ D_{1,1} \otimes E_{8,1}$	3^*
19	$\overline{qs_4}$	4	7	$D_{7,1}$	1
20	qn_4	4	5	$D_{5,1}$	1
			13	$D_{5,1} \otimes E_{8,1},\ D_{13,1}$	3^*
21	$\overline{qn_4}$	4	3	$A_{3,1}$	1
			11	$D_{11,1},\ A_{3,1} \otimes E_{8,1}$	1
22	qu_2	4	8	$D_{8,1}$	1
23	qv_2	4	4	$D_{4,1}$	1
			12	$D_{4,1} \otimes E_{8,1},\ D_{12,1}$	1
24	$qs_2 \otimes qs_2$	4	2	$A_{1,1}^{\otimes 2}$	1
			10	$D_{10,1},\ A_{1,1}^{\otimes 2} \otimes E_{8,1}$	1
25	$\overline{qs_2} \otimes \overline{qs_2}$	4	6	$D_{6,1}$	1
			14	$E_{7,1}^{\otimes 2},\ (A_{1,1}^{\otimes 2} \otimes D_{12,1})^+,\ D_{6,1} \otimes E_{8,1},\ D_{14,1},\ D_{1,1}^{\otimes 2} \otimes D_{12,1}$	3^*
26	$qs_2 \otimes \overline{qs_2}$	4	8	$A_{1,1} \otimes E_{7,1}$	1
27	$qs_2 \otimes LY$	4	19/5	$A_{1,1} \otimes G_{2,1}$	1
28	$\overline{qs_2} \otimes LY$	4	9/5	$A_{1,3}$	2
			49/5	$E_{7,1} \otimes G_{2,1},\ A_{1,3} \otimes E_{8,1}$	2

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Table B.5 – *Continued from previous page*

No.	$\mathcal{C}$	n	c	VOAs	Method
29	$qs_2 \otimes \overline{LY}$	4	31/5	$A_{1,1} \otimes F_{4,1}$	2
30	$\overline{qs_2} \otimes \overline{LY}$	4	21/5	$C_{3,1}$	2
31	$LY \otimes LY$	4	28/5	$G_{2,1}^{\otimes 2}$	1
32	$LY \otimes \overline{LY}$	4	8	$G_{2,1} \otimes F_{4,1}$	2
33	$\overline{LY} \otimes \overline{LY}$	4	12/5	$(A_{1,8})^+$	2
			52/5	$F_{4,1}^{\otimes 2}$, $(A_{1,8})^+ \otimes E_{8,1}$	2
34	$4fieldsx$	4	10/3	$(A_{1,1} \otimes A_{1,7})^+$	2
35	$\overline{4fieldsx}$	4	14/3	$G_{2,2}$	2

Appendix C

Source codes in the computations

We use the computer algebra software such as Magma and Mathematica in most of our computation.

C.1 Code and Lattice genera computation

We use the following source codes of Magma for computing the class number in each lattice genus. The idea is we construct the basic codes and apply the function **NumberGenus(C)** to get the class number in the genus of the lattices constructed from the code C. If Magma cannot compute the class number directly, we have to apply function **Lran(L,n)** to generate new isometric lattices in the genus until we get the result. The method is explained in chapter 4 and in Appendix A.

```
K := FiniteField(2);
Q := RationalField();
I1 := LinearCode<K,1 |[0]>;
function I(n)
   if n le 1 then
      return 1;
   else
```

```
        return ExtendCode(I1, n-1);
    end if;
end function;
function Code(C,D)
        return DirectSum(C,D);
   end function;

function CodeToLattice(C)
    L:=Lattice(C, "A");
    GM :=(1/2)*GramMatrix(L);
    return LatticeWithGram(GM);
end function;
function CodeLattice(C,D)
      L := DirectSum(C,D);
      return CodeToLattice(L);
end function;
function GenCompare(L,M)
    G1:=Genus(L);
    G2:=Genus(M);
    return G1 eq G2;
end function;
function NumberGenus(C)
     L := CodeToLattice(C);
     G := Genus(L);
     return #G;
end function;
d4 := LinearCode<K,4 |[1,1,1,1]>;
d6 := LinearCode<K,6 |[0,0,1,1,1,1],[1,1,1,1,0,0]>;
h8 := LinearCode<K,8 |[1,1,1,1,1,1,1,1]>;
e7 := LinearCode<K,7 |[0,0,0,1,1,1,1],[0,1,1,1,1,0,0],[1,0,1,0,1,0,1]>;
e8 := LinearCode<K,8 |[0,0,0,0,1,1,1,1],[0,0,1,1,1,1,0,0],
```

```
                            [1,1,1,1,0,0,0,0],[0,1,0,1,0,1,0,1]>;
d8 := LinearCode<K,8 |[0,0,0,0,1,1,1,1],[0,0,1,1,1,1,0,0],
                      [1,1,1,1,0,0,0,0]>;
d10 := LinearCode<K,10 |[0,0,0,0,0,0,1,1,1,1],[0,0,0,0,1,1,1,1,0,0],
                        [0,0,1,1,1,1,0,0,0,0],[1,1,1,1,0,0,0,0,0,0]>;
e15 := LinearCode<K,15  |[0,0,0,0,0,0,0,0,0,0,0,1,1,1,1],
                          [0,0,0,0,0,0,0,0,0,1,1,1,1,0,0],
                          [0,0,0,0,0,0,0,1,1,1,1,0,0,0,0],
                          [0,0,0,0,0,1,1,1,1,0,0,0,0,0,0],
                          [0,0,0,1,1,1,1,0,0,0,0,0,0,0,0],
                          [0,1,1,1,1,0,0,0,0,0,0,0,0,0,0],
                          [1,0,1,0,1,0,1,0,1,0,1,0,1,0,1]>;
d16 := LinearCode<K,16 |[0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1],
                        [0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0],
                        [0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0],
                        [0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0],
                        [0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0],
                        [0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0],
                        [1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0]>;
function Bran(B,n)
        for i in [1..n] do B[i] := Random(B);
        end for;
        v := B[1];
        for j in [2..n] do v +:= B[j];
        end for;
        return v;
end function;
function Lran(L0,n)
        B := Basis(L0);
        v := Bran(B,n);
        while Norm(v)  mod 3 ne 0 do  v:= Bran(B,n);
```

```
end while;
if Norm(v) mod 9 eq 0 then
        v1 := v;
else
        B1 := [b : b in Basis(L0) | (v,b) mod 3 ne 0];
        v -:= (Norm(v)*Modinv(2*(v,B1[1]),3) mod 9 )*B1[1];
        v1 := v;
end if;
        L100:= Neighbour(L0, v1, 3);
        G := Genus(L);
        bool := IsIsometric(L100, L);
        bool1 := IsIsometric(L100, L1);
        bool2 := IsIsometric(L100, L2);
        bool3 := IsIsometric(L100, L3);
        bool4 := IsIsometric(L100, L4);
        bool5 := IsIsometric(L100, L5);
        bool6 := IsIsometric(L100, L6);
        bool7 := IsIsometric(L100, L7);
        bool8 := IsIsometric(L100, L8);
        bool9 := IsIsometric(L100, L9);
        //bool10 := IsIsometric(L100, L10);
        //bool11 := IsIsometric(L100, L11);
        //bool12 := IsIsometric(L100, L12);
        //bool13 := IsIsometric(L100, L13);
        //bool14 := IsIsometric(L100, L14);
        //bool15 := IsIsometric(L100, L15);
        if      bool or bool1 or bool2 or bool3 or bool4
                or bool5 or bool6 or bool7 or bool8 or bool9
                then
                print "No";
        else
```

```
                    G100 := Genus(L100);
                    if G100 eq G then
                            print " in Genus 1";
                    else
                            print " in Genus 2";
                    end if;
                    print "======Yes======";
                    print "norm" , Norm(v1);
                    print "v :=", v1;
                    print "Kissing number =" , KissingNumber(L100);
            end if;
    return L100;
end function;
```

C.2 Fundamental matrix and VOA genus computations

To compute the fundamental matrix of the representation in chapter 5 we mainly use Mathematica in the computation. We also use Magma to compute the decomposition into irreducible representations.

Beginning with the set of conformal weights T_1 of the given MTC. We compute the exponent matrix L that is the matrix Λ. Next we compute the matrix A and characteristic matrix $\mathcal{X}$. Finally, we apply the function **Fm** to get the resulting fundamental matrix.

```
T1 = {0, h2, h3}
LN = DiagonalMatrix[Outer[Plus, T1]- c/24 // Flatten]
L = Mod[LN, 1]

A = Table[f[i, j], {i, 1, Length[L]}, {j, 1, Length[L]}]
list = Variables[A];
```

```
Reduce[{A.L.A == -17/18*A - 2*(A.L.L + L.A.L + (L.L).A) +
    3*(A.L + L.A) - 4*L.L.L + 8 L.L - 44/9*L +
    8/9*IdentityMatrix[Length[L]], A.A == A}, list, Backsubstitution -> True]

chi = Table[b[i, j], {i, 1, Length[L]}, {j, 1, Length[L]}]
list = Variables[chi];
Reduce[{31/36 (IdentityMatrix[Length[L]] - L) -
    1/864 (chi + L.chi - chi.L) == A}, list, Backsubstitution -> True]

Fm[L_, LD_, chi1_] := (ID = IdentityMatrix[Length[L]];
  NN = 10;
  del = q*Product[(1 - q^i)^24, {i, 1, NN}];
  E4 = 1 + 240*Sum[DivisorSigma[3, n]*q^n, {n, 1, NN}] + O[q]^NN;
  E6 = 1 - 504*Sum[DivisorSigma[5, n]*q^n, {n, 1, NN}] + O[q]^NN;
  JT = (E4^3)/del - 744;
  E10 = E4*E6;
  Ep = E10/del;
  DT = (1/Ep) ((JT - 240) (L - ID) + chi1 + L.chi1 - chi1.L);
  FFx = Table[Sum[b[i, j, k]*q^k, {k, -1, NN}], {i, 1, Length[L]}, {j, 1,Length[L]}];
  Do[If[Not[i == j], b[i, j, -1] = 0, b[i, j,-1] = 1], {i, 1,Length[L]},
  {j, 1, Length[L]}];
  FF = q^LD*FFx;
  diff3 = (q^(-LD)*(q*D[FF, q] - FF.DT)) // ExpandAll // Flatten // Normal;
  list = CoefficientList[diff3, q];
  erg = Solve[# == 0 & /@ list, Variables[list]];
  (FF /. erg[[1]]) + O[q]^(NN - 2);
  q^(-LD)*(FF /. erg[[1]]) + O[q]^(NN - 2))
```

Note that the parameter **LD** is the set of the diagonal entries of the exponent matrix.

We apply the following source codes to generate the possible affine Kac-Moody Lie algebras corresponding to the dimension and the central charge.

```
w = {Table[{n*(n + 2), n + 1}, {n, 1, 24}],
```

```
    Join[{{10000, 0}}, Table[{n*(2*n + 1), 2*n - 1}, {n, 2, 24}]],
    Join[{{10000, 0}, {10000, 0}}, Table[{n*(2*n + 1), n + 1}, {n, 3, 24}]],
    Join[{{1, 0}, {10000, 0}, {10000, 0}}, Table[{n*(2*n - 1), 2*n - 2}, {n, 4, 24}]],
    Join[{{10000, 0}, {14, 4}, {100000, 0}, {52, 9}, {1000000, 0}, {78,12}, {133, 18},
    {248, 30}}, Table[{10000000, 0}, {i, 9, 24}]]};

start[dim_, c_] := {{{}, {dim, c}}}

find[k_] := Module[{c, erg}, erg = {};
    dim = k[[2, 1]];
    c = k[[2, 2]];
    Do[If[(w[[i, j, 2]] + 1 >= dim/c) && (dim - w[[i, j, 1]] >= 0),
      Do[If[(w[[i, j, 2]] + l >= dim/c*l) && (dim - w[[i, j, 1]] >= 0),
          AppendTo[erg, {Join[k[[1]], {{i, j, l}}],
          {dim - w[[i, j, 1]],c - w[[i, j, 1]]*l/(w[[i, j, 2]] + l)}}]],
          {l, 1, If[{i, j} == {4, 1}, 1, 12]}]], {i, 1, 5}, {j, 1, c}]; erg];

Kandidat[l_] := Module[{s}, a = Select[l, #[[2, 1]] == 0 &]; b = Complement[l, a];
    Union[{Sort[#[[1]]], #[[2]]} & /@ Join[a, Flatten[find /@ b, 1]]]];

make[x_, y_] := Kandidat[Kandidat[Kandidat[Kandidat[Kandidat[Kandidat
[Kandidat[Kandidat[Kandidat[start[x, y]]]]]]]]]];
```

Remark: The result of the function **make**[dim,c] is of the form {{{{a,b,c}}, {d, e}}} where a is the "letter" type of the affine Kac-Moody Lie algebras (1 is for type "A", 2 is for type "B" etc.), b is the "rank" of the Lie algebras, c is the level, d is the remainder of the dimension, and e is the remainder of the central charge. For examples, {{{{1,1,1}}, {0, 0}}} is the Lie algebra $A_{1,1}$ and {{{{1,1,1},{5,7,1}}, {0, 1/2}}} is the Lie algebra (with its extension) $A_{1,1} \otimes E_{7,1}(1/2)$.

The following Magma source codes is used to decompose the representation into irreducible representations. The inputs are the S and T matrix of a given MTC and the results represent the irreducible components and the corresponding basis vectors. The following source codes correspond to the MTC qs_2. By changing S and T and using the appropriate cyclotomic filed, we will get the result for other MTCs.

```
F<zeta>:=CyclotomicField(24);
a := Sqrt(F!2);
S:=[1/a, 1/a,    1/a, -1/a];
T:=[zeta^(-1),0,   0,zeta^5];
G:=MatrixGroup< 2, F | S,T>;
M :=GModule(G);
fact:=DirectSumDecomposition(M);
[* [GModuleAction(x)(S),GModuleAction(x)(T)] : x in fact*];
[* [Morphism(x,M)]: x in fact*];
```

We use the following source codes from **Kac** to compute the simple current extensions.

```
Tensor
G A 1 5
G E 7 1
current a b
display
```

Remarks: In tensor mode, we can find the tensor product of Kac-moody Lie algebras by using the code: G [type] [rank] [level]. In the example above, it is the tensor product of $A_{1,5} \otimes E_{7,1}$. The command "current a b" computes a simple current where a and b are the numbers representing simple modules in the combination (a, b). Then we will get the result by using the command "display".